A General Equilibrium Open-Economy Model with Money, Endogenous Search, and Heterogeneous Firms

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Abstract

This paper develops a monetary general equilibrium model that features; (i) search frictions in the goods market, which create market power; (ii) endogenously chosen search effort by consumers, which mitigates this market power; (iii) heterogeneous firms and free entry; and (iv) an open economy, i.e. an arbitrary number of countries that trade goods and, potentially, assets. The paper is intended to lay the technical groundwork for future work, which will address issues such as the effects of monetary policy on production, firm entry, trade, and welfare.
1 Introduction

Models of endogenous search effort have recently found application to many macroeconomic questions, but have been strictly limited to closed economies. Furthermore, free entry of firms has been considered, but not in an environment with heterogeneous costs. However, allowing firm heterogeneity is important for three reasons: first, price dispersion is not always bad in this case; second, search effort has a direct welfare effect because it shifts production to the more efficient firms and increases TFP; and third, because the assumption of identical costs for all firms is incompatible with a monetary model of an open economy. Monetary policy will directly affect the cost of production, and differently so in each country. Homogeneity is therefore out of the window, and while it is possible to solve for equilibrium with a discrete cost distribution (for example one point of support for each country of origin), this comes with new technical problems. Full heterogeneity (a continuous cost distribution with unbounded support in each country) is actually the easier way to solve the model.

In this paper, I extend the optimal inflation model of Head and Kumar (2005) in two directions. First, I solve a version of the model with arbitrary distributions of costs (ex-ante firm heterogeneity), for the reasons discussed above. Second, I introduce international trade and propose a solution method for an arbitrary number of countries.

2 Literature

The debt to the predecessor of my model, Head and Kumar (2005), has already been acknowledged. Their model applies the price-posting mechanism of Burdett and Judd (1983) to the monetary search economy of Shi (1997), and studies the endogenous choice of search effort in steady state. Head, Kumar, and Lapham (2010) extend the model to cover aggregate nominal and real shocks.

Lagos and Wright (2005) developed an alternative microfounded model of money in a search economy, which has dominated applications of monetary search theory in recent years and is widely considered the benchmark model of a monetary search economy (e.g. Williamson and Wright, 2010). The problem for the purpose of this paper is that the quasi-linear production function in the “night market” of Lagos and Wright (2005), which makes the model tractable, would also pin down the real exchange rate between two countries (or with trade costs, make the real exchange rate indeterminate). But the behavior of the
real exchange rate in a monetary search setting is one of the targets of this project. The only paper I am aware of that studies exchange rates in a search setting is Head and Shi (2003), which is based on Shi (1997) and indeed predates Lagos and Wright (2005).

2.1 Endogenous search

Much of the monetary search literature studies a search effort decision by buyers, or an entry decision by buyers or sellers. In addition to Head and Kumar (2005) itself, the following papers need to be mentioned.

In an early contribution, Li (1995) introduced the “hot potato effect”: because buyers do not take the external effects of their search effort into account, a tax on money holdings which increases the rate of matching in general equilibrium can enhance welfare.\(^1\) Wright, Liu, and Wang (2009) study the effect further, but in contrast to this paper, they find that search effort decreases with inflation and only participation increases.

The externality due to endogenous search effort (or entry) is further studied by Shi (2006), Berentsen, Rocheteau, and Shi (2007), and Craig and Rocheteau (2008). In these papers, moderate inflation increases search effort and can mitigate the externality. Generally, the externality of endogenous search could be positive or negative; this arises from the bargaining mechanism that determines the terms of trade in their models. In this paper, by contrast, the terms of trade are determined by price posting, and the externality of search effort is always positive, i.e. more search effort puts pressure on firms to reduce markups, but individual buyers do not take this into account.

2.2 Some evidence

Recently, some papers have tested empirical implications of the Head and Kumar (2005) model. Caglayan, Filiztekin, and Rauh (2008) find that price dispersion is V-shaped as a function of inflation and that the amount of dispersion is related to search costs. Using European Union price data, Becker and Nautz (2010) and Becker (2011) find that this V-shaped relationship disappears in highly integrated markets, where search costs are presumably low.

\(^1\) In my paper, the velocity of money is fixed. But if anything, this makes my results stronger, as I find a similar effect with matching intensity.
In an unrelated literature, Meier (2010) studies inflation in countries undergoing persistent large output gaps (PLOGs). While the findings could be consistent with several explanations, they certainly do not contradict the predictions of the search model developed below. Consider for example Meier’s observation that “disinflation has tended to taper off at very low positive inflation rates, arguably reflecting downward nominal rigidities and well-anchored inflation expectations”, which is also consistent with a story in which expectations of slower money growth initially bring on disinflation, but then cause output to fall when expected money growth falls below the optimal level.

In this sense, the model in this paper may provide an alternative explanation for why output losses and/or sluggish growth would coexist with low inflation and interest rates, without appealing to sticky prices, and without implying liquidity traps or paradoxes of thrift. Further research certainly seems to be warranted.

3 Model

The model extends the optimal inflation framework introduced by Head and Kumar (2005). That framework was based on a monetary search economy (Shi, 1997) with price posting (Burdett and Judd, 1983) and endogenous search effort by buyers. To summarize what follows, the innovations of this paper are to:

- Introduce a new matching process, inspired by Mortensen (2005) but slightly different (see appendix A).
- Introduce firms and consider free entry.
- Re-derive the model where all sellers/firms have the same costs. With the new matching process, a unique equilibrium exists under weak assumptions, and can be expressed in closed form.
- Solve the model for heterogeneous firms, with and without free entry. At this point, numerical solutions become necessary.
- Solve a multi-country model with trade.
3.1 Environment

Time is discrete and infinite. There is a measure 1 of households. Members can be either shoppers or workers; there is a positive measure of each type, and they sum to measure 1. There exists a government that supplies a stock $M_t$ of perfectly durable, infinitely divisible asset called money at time $t$, and augments this stock at the beginning of each period with lump-sum transfers $T_t$ to the households. Household members do not have independent utility but share equally in the utility of the household (Shi, 1997). Households value the streams of consumption $c$, search effort $s$, and work effort $n$ according to the separable utility function:

$$U(c, s, n) = (1 - \beta) \sum_{t=0}^{\infty} \beta^t [u(c) - \mu s_t - n_t]$$

(1)

Assume $\beta \in (0, 1)$ and $\mu > 0$, with the disutility of labor normalized to 1. $u(\cdot)$ is strictly increasing, strictly concave, and satisfies the Inada conditions. Furthermore, $u'(c)c$ is non-increasing in $c$.

So far, the set-up is similar to that of Head and Kumar (2005), but for concreteness I will additionally assume that there exists a measure $N_t$ of firms, owned by households, that hire workers in a perfectly competitive labor market in order to produce goods. In order to trade these goods, firms and shoppers enter an anonymous and memoryless goods market characterized by search frictions. The infinitely many shoppers of a household must search separately and, once in the market, cannot coordinate with their siblings.

The trading period proceeds as follows: Firms learn their costs, as well as all macroeconomic variables, and post prices. As they can perfectly forecast demand (by the law of large numbers applied to the mass of shoppers), they then hire workers and produce. Households learn the price distribution and decide how much money $m_t$ each member carries into the market, and how much effort $s$ should be spent on obtaining price offers. Once in the market, each shopper receives $k$ random quotes (Burdett and Judd, 1983), i.e. is able to observe the prices of $k$ firms, where $k$ is drawn from a distribution $Q(k|s)$ with support $\{1, 2, \ldots\}$. The distribution of quotes with higher search effort strictly first order stochastically dominates the one with lower search effort: informally, more effort supplies more quotes. The precise set of assumptions sufficient for the results of this paper is stated in appendix A.

The shopper can then purchase goods from any of the firms whose prices he has observed,
but he cannot spend more money than he carries. To keep the analysis tractable, shoppers
cannot coordinate with one another, cannot spend any extra effort after receiving quotes,
and cannot recall any quotes from a previous period. Similarly, shoppers remain anony-
mous to firms, so firms will only accept payment in cash. At the end of the period, firms
pay their workers and remit the profits to their owners. Workers take home their pay,
shoppers take home their goods, and the members of the household share equally in con-
sumption and earnings.

To simplify notation, all monetary variables will be expressed in “constant-money” terms,
which means the nominal value divided by the money stock $M_t$. I avoid the term “real” in
order to prevent confusion, because “real” is commonly understood to mean the nominal
value divided by the price level $P_t$. In this paper, money has nonmonotonic effects even in
steady-state, so dividing by the money stock is very different from dividing by the price
level, but necessary to keep all variables stationary.

### 3.1.1 Matching

Let $q_k(\eta), \ k = 1, \ldots, K$ be the probability that a shopper who searches with success rate
$\eta$ observes exactly $k$ prices. Assume that $\eta = s \cdot N$, where $s$ is search intensity and $N$ is
the mass of active firms; it seems reasonable that matching success should depend both on
shopper effort and on firm presence. (Through this channel, firm entry will put downward
pressure on prices.) Define the functions $J : [0, 1] \times [0, \infty) \rightarrow [0, 1]$ and $a : [0, 1] \times [0, \infty) \rightarrow
[0, \infty)$:

$$J(F, sN) = \sum_{k=1}^{K} q_k(sN) (1 - (1 - F)^k)$$

$$a(F, sN) = \sum_{k=1}^{K} q_k(sN) k (1 - F)^{k-1}$$

Let the cumulative distribution function of prices posted by sellers be $F_t(p_t)$ on support
$F_t$. Naturally, shoppers who observe more than one price will only buy from the cheapest
firm. Then the c.d.f. of the transactions prices is the c.d.f. of the lowest price observed by
a buyer, which happens to be

$$J(F_t(p_t), s_tN_t) \ \forall p_t \in F_t.$$
Similarly, how many transactions can a firm with price $p_t$ expect when all shoppers search with success rate $sN$? The answer is

$$J_1(F_t(p_t), s_tN_t) = a(F_t(p_t), s_tN_t) \quad \forall p_t \in F_t,$$

where the subscript $J_1$ denotes the derivative with respect to the first argument. For this reason, I call $a(F, sN)$ the arrival function in analogy to the arrival rates common in search theory.

Throughout this paper, quantitative results are derived using the geometric matching process $q_k(\eta) = \frac{\eta^{k-1}}{(1+\eta)^k}$ with upper limit $K = \infty$ and expectation $\bar{q} = 1 + \eta$. Intuitively, the process can be characterized by the shopper flipping a coin (loaded by $\eta$) until it comes up tails, and then getting one quote for each flip. Appendix A discusses other reasonable matching processes, including those used in the literature.

### 3.1.2 Households

Households take the distribution of prices $F_t(p_t)$ and the wage $w_t$ (all expressed in constant-money terms) as given, and choose the search effort and expenditure strategies for its shoppers, as well as the work effort of its workers. As the sub-utility of consumption $u(c)$ is strictly concave, the household will treat all of its shoppers the same. Therefore, all shoppers pursue the same expenditure strategy $x_t(p_t)$ at time $t$. With household money stock $m$ and aggregate money stock $M$, the household optimizes

$$v(m, M) = \max_{m', x(\cdot), c, s, n} \left\{ u(c) - \mu s - n + \beta \mathbb{E} \left\{ v(m', M') | M \right\} \right\}$$

subject to

$$\frac{m'}{m} = 1 - \int_{F} \frac{x(p)}{m} a(F(p), sN) \, dp + \Pi(M) + wn + \frac{T}{m}$$

$$c = \int_{F} \frac{x(p)}{p} a(F(p), sN) \, dp$$

where $\Pi(\cdot)$ denotes aggregate profits.

Denote the Lagrange multiplier on the budget constraint (4) by $\Omega_t$. (The budget constraint is expressed in constant-money terms to make $\Omega_t$ stationary.) As discussed in section 3.1.3, a shopper carrying $m_t$ units of currency cannot buy more than $\frac{m_t}{M_t p_t}$ of goods at nominal price $M_t p_t$ ($p_t$ in constant-money terms). Because all buyers share equally in their house-
hold’s utility and are too small to influence consumption \( c_t \), they will pursue a reservation price strategy, such that:

\[
x_t(p) = \begin{cases} 
m_t & \text{if } p \leq \bar{p}_t \\
0 & \text{otherwise} 
\end{cases}
\]  

(6)

The key constraint is that once in the market, they cannot transfer the money to another shopper. Going home means leaving the money idle until next period; and the value of idle money at the end of period \( t \) is \( M_t \Omega_t \), hence the nominal reservation price satisfies:

\[
M_t \bar{p}_t = M_t \frac{u'(c_t)}{\Omega_t}
\]  

(7)

The labor supply is perfectly elastic: households will meet any demand for labor as long as the wage satisfies \( w_t \geq 1/\Omega_t \).

Let \( c(s) \) denote the consumption resulting from searching with effort \( s \). By the law of large numbers, this is a deterministic function. If it is also concave, then the optimal choice of search effort implies that \( u'(c(s))c'(s) = \mu \).

Finally, the key intertemporal variable, \( \Omega \), is determined by the first-order condition for \( m_{t+1} \) together with the envelope condition for \( m_t \), iterated by one period:

\[
\Omega_t = \mathbb{E}_t \left\{ \beta m_t \frac{u'(c_{t+1})c_{t+1}}{m_{t+1}} \right\}.
\]  

(8)

### 3.1.3 Firms

Let there be a mass \( N \) of firms. In this section, suppress time subscripts, and write \( a(F, sN) \) as simply \( a(F) \).

Assume that a firm requires \( \phi \) units of labor to produce one unit of output. In a perfectly competitive labor market, firms can hire any quantity of labor at the nominal wage \( Mw \). With nominal price \( Mp \), they will attract \( a(F(p)) \) shoppers (the price distribution \( F(p) \) is expressed in constant money terms so as to be stationary). If \( p \leq \bar{p} \), each shopper will spend all the money they carry, and as shown above, all shoppers carry the same amount of money, \( M \). Therefore, the nominal profits of a firm with marginal cost \( \phi \) are determined by:

\[
M \pi(\phi) = \max_p \left\{ M \left( 1 - \frac{\phi w}{p} \right) \frac{a(F(p))}{N} \mid p \leq \bar{p} \right\}.
\]  

(9)

As shown in Head and Kumar (2005), when all firms have the same marginal cost \( \phi \), they must all make the same profits; in particular, the same profits as any firm which charges
the reservation price. This allows us to solve

\[ a(F(p)) = a(1) \frac{1 - \frac{\phi w}{\bar{p}}}{1 - \frac{\phi w}{p}}, \] (10)

for the price distribution \( F(p) \) if we know \( a(\cdot) \).

When firms have heterogeneous costs, however, clearly they do not all make the same profits. The firm’s problem must be solved in the following way: assume that each firm draws a value of \( \phi \), the marginal labor cost of production, from a distribution \( G(\phi) \) with support \( \mathbb{G} \subset \mathbb{R} \) and upper bound \( \bar{\phi} \). (The nominal marginal cost of production is then \( \phi Mw \).) Abusing notation, \( \phi \) will sometimes identify a representative seller with disutility \( \phi \).

**Lemma 3.1.** Assume \( p_1 \) and \( p_2 \) are solutions to profit maximization with \( \phi_1 \) and \( \phi_2 \), respectively (we do not require them to be unique solutions). If \( \phi_1 < \phi_2 \), then \( p_1 < p_2 \).

**Proof.** See appendix B.

Consequently, firms’ prices are completely ranked by their costs. If the support of \( G \) includes \( \bar{p}/w \), i.e. \( \bar{\phi} w > \bar{p} \), some firms (with mass \( 1 - G(\bar{p}/w) \)) will not be able to sell their products for any profit at all. In particular, I assume that \( \bar{\phi} = \infty \) in all quantitative applications. To simplify matters, I assume that all such noncompetitive firms are simply inactive, and are ready to produce as soon as the reservation price rises. Define the cost distribution conditional on producing:

\[ \tilde{G}(\phi) = \frac{G(\phi)}{G(\bar{p}/w)} \] (11)

Assuming that \( G \) is differentiable, guessing that \( F(p) \) is differentiable, and guessing that the solution to profit maximization is a differentiable function which we denote by \( p(\phi) \), this implies the ranking conditions:

\[ F(p(\phi)) = \tilde{G}(\phi) \text{ for } \phi w \leq \bar{p} \] (12)

\[ F'(p(\phi))p'(\phi) = \tilde{G}'(\phi) \] (13)

\(^3\) Beware the apparent similarity with Melitz-type monopolistic competition models. In them, firms are often identified by their productivity \( \varphi \), but that is the inverse of marginal cost.
Profit maximization implies the following first-order condition:

\[ \frac{\partial \pi}{\partial p}(p; \phi) = \left( 1 - \frac{\phi w}{p} \right) a_1(F(p))F'(p) + \frac{\phi w a(F(p))}{p^2} \frac{N}{N_F} \]

(14)

Together, (12), (13), and (14) determine a differential equation

\[ p'(\phi) = \left( \frac{p(\phi)^2}{\phi w} - p(\phi) \right) \left( \frac{-a_1(\tilde{G}(\phi))}{a(\tilde{G}(\phi))} \right) \tilde{G}'(\phi) \]

(15)

with boundary condition

\[ p(\bar{p}/w) = \bar{p}. \]

(16)

Equation (15) is a special case of the Riccati equation, which has the following solution.

**Lemma 3.2.** Let \( p(\phi) = \infty^4 \) if \( \phi w > \bar{p} \) and

\[ p(\phi) = \frac{a(\tilde{G}(\phi))\bar{p}}{a(1) - \frac{\bar{p}}{w} \int_\phi^{\bar{p}/w} a'(\tilde{G}(t)) \bar{G}'(t) \, dt} \]

(17)

otherwise. Then \( p(\phi) \) solves the system \{(15), (16)\} and therefore maximizes profits (9).

**Proof.** See appendix B.

### 3.2 Equilibrium in autarky without free entry

Even without free entry, the mass of active firms can potentially vary with the reservation price. Denote the number of potential firms by \( \bar{N} \), and the mass of active firms by \( N = \bar{N}G(\bar{p}/w) \) in the case where the firms have different marginal costs. Define the gross money growth rate \( \gamma_t = 1 + \frac{T_t}{M_t} \), and assume that the stochastic process of \( T_t \) (and therefore \( M_t \)) is such that \( \gamma_t \) is stationary.

**Definition 1.** A stationary monetary search equilibrium (SMSE) is a collection of a value function \( v(m, M) \), policy functions \( m'(m, M) \), \( x(\cdot; m, M) \), \( s(m, M) \), \( n(m, M) \), common expenditure rule, \( X(\cdot; M) \), common search effort \( S(M) \), a wage \( w(M) \), a distribution of posted prices, \( F(\cdot; M) \), and expectations \( \gamma'(M) \) such that

\[ \]
1. The value function \( v(m, M) \) solves (3) with the associated policy functions \( m'(m, M) \), \( x(\cdot; m, M) \), \( s(m, M) \), and \( n(m, M) \), and Lagrange multiplier \( \Omega(M) \).

2. The price distribution \( F(\cdot; M) \) satisfies (10) in case all firms have the same marginal cost, or (12) together with (17) otherwise.

3. Aggregate (constant money) profits in (4) are \( \Pi(M) = \left( 1 - \frac{\phi w(M)}{p(M)} \right) a(1, SN) \) in case all firms have the same marginal cost, or
   \[
   \Pi(M) = \int_{\mathcal{X}} \left( 1 - \frac{\phi w(M)}{p} \right) a(F(p; M), S(M)N) \, dp
   \]
   otherwise.

4. The money market clears: \( w(M)n(M, M) + \Pi(M) = 1 \). By Walras’ Law, this is equivalent to labor market clearing.

5. Individual choices equal aggregate quantities: \( x(p; M, M) = X(p; M) \) for all \( p \), \( s(M, M) = S(M) \), and \( m'(M, M) = \gamma M \).

6. Expectations \( \gamma' | M \) are correct.

7. Money has value: \( F(p) < 1 \) for some \( p < \infty \).

Additional uncertainty about parameters is easy to incorporate, as \( \Omega_t \) fully summarizes all intertemporal expectations and all utility and cost parameters only affect the equilibrium contemporaneously. In the aggregate, we can write:

\[
\Omega_t = \mathbb{E}_t \left\{ \frac{\beta}{\gamma_{t+1}} u'(c_{t+1}) c_{t+1} \right\}.
\]

(18)

As the labor market is perfectly competitive, the wage must just compensate workers for their work effort:

\[
w = \frac{1}{\Omega}.
\]

(19)

### 3.2.1 Fixed search

Equation (5) can be combined with the optimal pricing formula to express aggregate consumption as a function of aggregate search effort \( s \) (more details in appendix B.1). Consider first the case where all firms have the same marginal cost \( \phi \), which provides valuable
intuition:
\[ c(s) = \frac{1 - a(1, sN)}{\phi w} + \frac{a(1, sN)}{\bar{p}}, \]

Note that \( a(1, \cdot) \) equals \( q_1(\cdot) \), i.e. the fraction of consumers who receive a single quote. In this sense, the market outcome is equivalent to a world where prices are subject to Bertrand-style negotiation. If shoppers who had more than one quote were able to bring the price down to the efficient level \( \phi w \), while all those that didn’t had to accept their reservation price \( \bar{p} \), the market outcome would be the same. However, this intuition does not carry through to the optimal choice of search effort: the function \( c(s) \) is derived for a symmetric equilibrium where all shoppers choose the same amount of search effort. As no single shopper can affect the search effort of their fellow shoppers, they will typically not choose a socially optimal amount of effort.

Substituting (7) and (19), and rearranging, we derive the market equilibrium equation:
\[ u'(c)c = \Omega \left( \frac{u'(c)}{\phi} (1 - a(1, sN)) + a(1, sN) \right) \]

In order to obtain a relationship in \((s, c)\)-space, we have to replace \( \Omega \). There are three useful ways of doing this.

**Fixed expectations:** When expectations of future consumption and money growth are fixed, the curve in \((s, c)\)-space is downward sloping for low \( \Omega \), then vertical, and finally upward sloping for high \( \Omega \). It shifts rightward with increasing \( \Omega \) (higher \( s \) for any given \( c \)).

**Steady state:** Assume that there is no uncertainty and \( c_{t+1} = c_t \). Then equation (21) becomes:
\[ \frac{\gamma}{\beta} = \frac{u'(c)}{\phi} (1 - a(1, sN)) + a(1, sN). \]

This curve is upward sloping in \((s, c)\)-space, and shifts right/down for higher money growth \( \gamma \). Therefore, higher money growth implies lower consumption for a fixed level of search effort. Consequently, as long as search effort is fixed above zero, the optimal rate of steady-state money growth satisfies \( \gamma = \beta \) (the “Friedman rule”). The right-hand side achieves its minimum of 1, which implies the first-best \( u'(c) = \phi \).

**Nominal interest rates:** It is easy to introduce nominal risk-free bonds into the household’s problem (4). The nominal interest rate can be found assuming zero net supply
of these bonds:

\[
1 + i_t = \frac{u'(c_t)c_t}{\Omega_t} = \frac{u'(c_t)c_t}{\mathbb{E}_t \left\{ \frac{\beta}{\gamma_{t+1}} u'(c_{t+1})c_{t+1} \right\}}
\]

Equation (21) is then equivalent to

\[
i = \left( \frac{u'(c)}{\phi} - 1 \right) (1 - a(1, sN)).
\]

This expression has several advantages. First, it is more satisfying than fixed expectations but does not require a steady state, or indeed certainty, because the nominal interest rate summarizes all expectations about the future. Secondly, it demonstrates how high nominal interest rates imply low consumption (market power) and/or higher search (shoe-leather costs). Thirdly, any policy that achieves \( i = 0 \) (an alternative statement of the Friedman rule) implements the first-best \( u'(c) = \phi \), for any \( s \) fixed above zero. Lastly, it evokes the cash-in-advance ancestry of the model and the implication that \( i < 0 \) is inconsistent with any equilibrium.

Consider now the case where firms have different marginal costs. Denote the infimum of the support of \( G(\cdot) \) by \( \underline{\phi} \). The upper bound is given by the reservation cost \( \bar{p}/w \), which we can now see to equal \( u'(c) \). The number of active firms is therefore \( N(c) = N G(u'(c)) \). The market equilibrium equation becomes:

\[
u'(c)c = \Omega \left[ a(1, sN(c)) + \int_{\underline{\phi}}^{u'(c)} \frac{u'(c)}{t} \left( -a_1(\tilde{G}(t), sN(c)) \right) \tilde{G}(t) \tilde{G}'(t) \, dt \right]
\]

The shape of this curve in \((s, c)\)-space is much the same as in the simpler case above. The complication arises through the endogenous reservation price, which implies that not all firms will be able to compete. However, this complication is not very severe, especially when the cost distribution \( G(\phi) \) is well-behaved and does not put too much weight on the tails. In steady-state terms:

\[
\frac{\gamma}{\beta} = a(1, sN(c)) + \int_{\underline{\phi}}^{u'(c)} \frac{u'(c)}{t} \left( -a_1(\tilde{G}(t), sN(c)) \right) \tilde{G}(t) \tilde{G}'(t) \, dt
\]

Outside of a steady state, the nominal interest rate analog of (24) is obtained by setting \( 1 + i \) equal to the RHS of (26). The socially optimal monetary policy is still the Friedman
rule, but the associated level of consumption is now harder to compute.

3.2.2 Optimal search

As discussed in the previous section, the socially optimal monetary policy is the Friedman rule when search effort is fixed, because the Friedman rule minimizes firms’ market power. This result still holds when search effort is socially chosen, by maximizing $u(c) - \mu s$ subject to the constraint (21) or (25).

However, the result breaks down when households privately choose their search effort. The reason is that search effort accomplishes two things: it allows shoppers to pick up low-price offers, and through that, it constrains the market power of the firms. The key is that because each household is small, only the former effect is internalized. As a result, loose monetary policy creates price dispersion and forces shoppers to search harder.

Solving the households’ problem (equation (3)), privately optimal search implies that $u'(c(s))c'(s) = \mu$, taking the distribution of prices $F(p)$ as given. Details of the derivation are in appendix B.2; we need to define the auxiliary function

$$h(F, \eta) = \int_0^F \frac{a_2(z, \eta)}{a(z, \eta)} \, dz$$

(the index $a_2$ refers to the derivative with respect to the second argument), and can then derive the optimal search equation for the case where all firms have the same cost:

$$\mu = \Omega N \left( -a(1, sN)h(1, sN) \right) \left( \frac{u'(c)}{\phi} - 1 \right)$$

We can use the market equilibrium equation (21) to divide out $\Omega$ and obtain a purely contemporaneous relationship:

$$\mu = N u'(c)c \frac{-a(1, sN)h(1, sN) \left( \frac{u'(c)}{\phi} - 1 \right)}{u'(c) \left( 1 - a(1, sN) \right) + a(1, sN)}$$

The behavior of the denominator and of the markup $\left( \frac{u'(c)}{\phi} - 1 \right)$ is quite monotonic. If furthermore $u'(c)c$ is non-increasing in $c$,\footnote{Head and Kumar (2005) make this assumption throughout.} the behavior of equation (29) in $(s, c)$-space is driven by the term $-a(1, sN)h(1, sN)$.
Recall first that $a(1, \cdot)$ equals $q_1(\cdot)$, i.e., the fraction of consumers who receive a single quote. This fraction is a strictly decreasing function of search effort, and it seems reasonable to assume that it approaches zero as search effort becomes infinite. Also note that $h(1, s) < 0$ for $s > 0$ and $h(1, s) = 0$ as $s = 0$, and it is continuous in $s$ (proofs in appendix B), so it must decrease for low levels of search effort. Consequently, the term $-a(1, sN)h(1, sN)$ is positive for $s > 0$ and zero for $s = 0$; it must therefore strictly increase for low levels of search effort, and then declines back to zero for high levels of search effort (lemma B.1 in appendix B). As a result, equation (29) describes an inverted u-shaped curve in $(s, c)$-space. This curve is purely contemporaneous (unaffected by expectations) and reflects both market equilibrium and the optimal choice of search effort.

If, $u'(c)c$ was increasing in $c$, the behavior of equation (29) could be strange. For example, with $u(c) = \frac{c^2 - \sigma - 1}{1 - \sigma}$ and $\sigma < 0.5$, the equation describes circles in $(s, c)$-space, concentric for increasing values of the disutility of search effort, $\mu$. When $\mu$ is too high, no real solution for $(s, c)$ exists. If $\sigma \in (0.5, 1)$, the behavior is intermediate; for low levels of money growth $\gamma$, no steady state equilibrium exists, and for higher levels of $\gamma$, there may be multiple equilibria. I am not sure what to make of this case.

Going back, we can state:

**Theorem 3.1.** If the following conditions hold then a SMSE $\{(\Omega_t, c_t, s_t)\}_{t=0}^{\infty}$ exists, is unique, and is fully described by (18), and by (21) and (29) for each time period $t$.

$E1$. The matching process satisfies assumptions M1–6 in appendix A.

$E2$. The sub-utility of consumption is such that $u'(c)c$ is strictly decreasing in $c$.

$E3$. The money growth process $\gamma_t$ is stationary and such that $\Omega < u'(c)c$ in all states and all time periods.

**Proof.** (Sketch for the steady state; a full proof is given in appendix B.) By condition (E2), the solution to equation (29) approaches the $c$-axis faster than the solution to equation (22) for any $\Omega \leq u'(c)c$. The solution to (22) is strictly increasing in $(s, c)$-space throughout, approaching $(s \to \infty, c \to c^*)$ (let $c^*$ be the efficient level of consumption given by $(u')^{-1}(\phi)$). By condition (E1), the term $-a(1, sN)h(1, sN)$ is positive and inverted u-shaped. Therefore, the solution to (29) achieves a maximum $\hat{c}$ below $c^*$ and approaches $(s \to \infty, c \to 0)$. \[\Box\]

Conditions E1 and E2 could be relaxed, if desired, for fortuitous combinations of parameters.
For suitable matching processes (see appendix A), this optimal choice of search effort is a stable interior equilibrium.\(^6\)

Again, consider now the case where firms have different marginal costs. Recall that \(\phi\) is the infimum of the support of \(G(\cdot)\), that the upper bound is given by the reservation cost \(\bar{p}/w = u'(c)\), and that the number of active firms \(N(c) = \bar{N}G(u'(c))\) is declining in \(c\). The optimal search equation becomes:

\[
\mu = \Omega N(c) \left[ a(1, sN(c))h(1, sN(c)) + \int_{\phi}^{u'(c)} \frac{u'(c)}{t} \left( -a_1(\bar{G}(t), sN(c)) \right) h(\bar{G}(t), sN(c)) \bar{G}'(t) dt \right]
\]

As was the case for the market equilibrium equation, the optimal search equation is complex in the case of heterogeneous firms, but it behaves materially the same as in the simpler case of common costs, and looks the same in \((s, c)\)-space after substituting out \(\Omega\) using (25). Under the same sufficient conditions, with two additions, a SMSE \(\{(\Omega_t, c_t, s_t)\}_{t=0}^{\infty}\) exists, is unique, and is fully described by (18), (25), and (30) for each time period \(t\). The additional conditions require that the cost distribution \(G(\phi)\) is well-behaved: (1) it does not put too much weight on the tails, and (2) it does not have multiple modes. In the latter case, existence is preserved, but there may be multiple equilibria.

### 3.2.3 Shocks

The three-equation system of theorem 3.1 is well-suited to intertemporal analysis in the spirit of a real or monetary business cycle model. The equations could be Taylor approximated around the steady state, which would yield the policy and impulse response functions in the usual way.

### 3.2.4 Free entry

Consider the case when all firms have the same cost. Say that the entry is subject to a fee \(k_E < 1\), assessed in constant-money terms\(^7\) and each period.\(^8\) Assume that this entry fee is

---

\(^6\) This was not the case in the original model of Head and Kumar (2005). The point is that consumption as a function of search effort, taking prices as given, is strictly concave for certain matching processes. At this point this is a conjecture, but it is easy to verify graphically in those cases.

\(^7\) The qualitative results are the same if the cost is assessed in labor terms.

\(^8\) Like in Melitz (2003), this simplifies the analysis, but ignores the effect of interest rates on funding costs.
distributed among households, i.e. added to equation (4). Firms will enter if their profits exceed the entry fee, and exit otherwise. This yields the free-entry condition:

\[ N k_E = \Pi = \left( 1 - \frac{\phi}{u'(c)} \right) a(1, sN) \]  

(31)

The SMSE is now expressed as \[ \{(\Omega_t, c_t, s_t, N_t)\}_{t=0}^{\infty}, \] which solve equations (18), (21), (29), and (31) for each \( t \). Dividing equation (29) by equation (31) yields a curve in \((sN, c)\)-space that takes the role of equation (29) in the previous discussion. The market equilibrium equation (21) depends only on \( sN \), not on \( s \) or \( N \) directly. The only difference is that the new optimal search curve is a bit flatter than previously, but it has the same shape, and the SMSE exists and is unique under the same conditions as before, with the addition of \( k_E < 1 \) to ensure that some firms do enter.

A similar analysis can be conducted in the case where firms have heterogeneous costs. Define the function \( \Gamma(\phi) = \int_0^\phi t \bar{G}(t) \, dt \) (the conditional mean below \( \phi \)). Aggregate profits are, in constant-money terms (derivation in appendix B.5):

\[ \Pi = 1 - a(1, sN(c)) \frac{\Gamma(u'(c))}{u'(c)} - \int_0^\phi \frac{\Gamma(t)}{t} \left( -a_1(\bar{G}(t), sN(c)) \right) \bar{G}'(t) \, dt \]  

(32)

The rest follows from \( N(c) = \bar{N}G(u'(c)) \), setting \( \Pi = N(c)k_E \), and making \( \bar{N} \) an endogenous variable. Again, the nature of equilibrium does not change materially.

### 3.2.5 Comparative statics

For details and proofs relating to this section, see appendix B.4.

What are the effects of real shocks on the steady-state equilibrium? The effect of an increase in the production cost \( \phi \) is a fall in consumption and a rise in search effort. On the other hand, an increase in the disutility of search, \( \mu \), leads to a fall in both consumption and search effort. The same result holds when firms have heterogeneous costs, if we define a cost shock in the first-order stochastic dominance sense: If \( G_2(\phi) < G_1(\phi) \) for all \( \phi \), then \( G_2 \) implies lower consumption and higher search effort than \( G_1 \) in steady state. Thirdly, an increase in the cost of firm entry lowers consumption, and generally reduces search effort.

Whereas the three cost parameters above represent exogenous shocks to the economy, the government probably has a degree of control over the money growth process \( \{\gamma_t\} \). How-
ever, only *expectations* of future money growth affect the value of money, Ω, and therefore the real economy; surprise money growth shocks that do not affect expectations have no real effect at all in this model (prices are fully flexible, money is neutral). Therefore, I focus on steady states with constant and expected money growth γ each period. As shown in appendix B.4, there exists a consumption-maximizing rate of money growth ˆγ > β. Consumption is increasing in γ for β < γ < ˆγ and decreasing in γ otherwise. Search effort is increasing in γ throughout. With free entry, the number of firms is decreasing in γ.

The effects of β are simply inverse to those of γ. Furthermore, when there is uncertainty, all expectations of future money or consumption growth can be neatly summarized by the nominal interest rate: $1 + i$ replaces $\gamma/\beta$ in the previous discussion.

If we define the aggregate nominal price level by

$$P_t = \frac{M_t}{c_t},$$

(33)

we can also discuss the effects of money growth on inflation. Clearly, in the long run, money growth must equal inflation (after correcting for stable consumption growth). In the short run, however, they may be quite separate, because of the effect of money on consumption growth. Consider a state where expectations of money growth are so low that consumption would increase with higher money growth. A negative shock to money growth could reasonably decrease expectations of further money growth. In this case, the negative shock to money growth also causes a decline in consumption. By equation (33), price inflation then falls by less than money growth, and may even rise in rare cases. This thought experiment emphasizes that money growth γ, and shocks to it, are not properly thought of as representing “inflation”. While in the long run, the “optimal rate of money growth” implies an “optimal rate of inflation”, this is not true for the short run. Consequently, the inflation environment may not be a reliable guide to the money growth environment and its effects on consumption. Monetary aggregates or nominal GDP growth may be better candidates.

Lastly, it is worth paying attention to the interaction between the search cost µ and the optimal rate of money growth in steady state. A higher µ shifts the optimal search curve down in (s, c)-space, implying a lower level of consumption for a given search effort. Assume that γ was previously such that the consumption maximum was achieved. As the market equilibrium curve slopes up for any level of γ, the new consumption maximum is to the right and below the formerly optimal market equilibrium curve. Consequently, the optimal rate of inflation is increasing in the search cost µ; and this is also true for the cost
of firm entry in the model with free entry of firms.

3.3 The open economy

Let there be two countries, Home and Foreign, and denote Home variables with the subscript $H$ and Foreign variables with the subscript $F$. Define the nominal exchange rate $E$ in terms of Home currency divided by Foreign currency, and define the real exchange rate using the Home and Foreign price levels: $\varepsilon = \frac{E P_F}{P_H}$ (in terms of Home consumption divided by Foreign consumption). By equation (33), this implies:

$$\varepsilon = E \frac{M_F c_H}{M_H c_F}$$

(34)

The nominal and real exchange rates are of most practical interest, but the model is much easier to solve in terms of the wage-based exchange rate (in terms of Home labor divided by Foreign labor; or utility, since the disutility of labor was normalized to 1), because this is the exchange rate relevant for firms’ competition. Define

$$v = E \frac{M_F w_F}{M_H w_H} = \varepsilon \frac{c_F \Omega_H}{c_H \Omega_F}.$$  

(35)

Trade proceeds as follows. Firms can advertise prices in either country, but in local currency. From the point of view of the consumer, domestic and imported goods are indistinguishable. There is also no bias on how likely a consumer is to observe a domestic or importing firm. Whenever a consumer likes the price and wants to purchase the good, the firm produces the good and ships it to the buyer. Of course, goods cannot be shipped costlessly. Trade costs are assessed in the familiar iceberg form: in order to sell $x$ units of a good in a foreign market, a firm needs to ship $\tau x$ units from home, with $\tau > 1$. So while there is no bias against imported goods, domestic firms are able to charge lower prices on average than the importing competition.

There is, however, a serious technical problem with the case where all firms in a country have the same costs. Even if domestic and foreign competitors were to have the same labor requirement in each of their home countries, they cannot possibly have the same costs in the market, as real wages will vary with the state of the economy, even without any trade costs. It is not too difficult to solve for equilibrium when the cost distribution has two points of support - domestic and foreign firms - but the real problem is that this
equilibrium would feature seven (!) separate regimes: one in which foreign importers have lower costs than home producers, one vice versa, one where importers have higher costs than domestic producers in each country, and two where firms in one country cannot even compete at the other country’s reservation price, plus four borderline cases. This is inconvenient as it is, and completely prohibitive for more than two countries. As a result, it makes more sense to proceed with heterogeneous firms from here on.

In order to simplify the analysis for now, I assume that the cost distribution of domestic firms is the same in each country, and that any firm can export if is able to compete abroad. Assume that the mass of potential firms is \( N_H \) at Home and \( N_F \) in the Foreign country. The distribution of active firms selling in either country is therefore:

\[
\tilde{G}_H(\phi) = \frac{N_H}{N_H + N_F} \frac{G(\phi)}{G(u'(c_H))} + \frac{N_F}{N_H + N_F} \frac{G(\frac{\phi}{v})}{G(v't)} \tag{36}
\]

\[
\tilde{G}_F(\phi) = \frac{N_H}{N_H + N_F} \frac{G(\frac{\phi}{c_H})}{G(u'(c_F))} + \frac{N_F}{N_H + N_F} \frac{G(\phi)}{G(u'(c_F))} \tag{37}
\]

As before, the distribution of quotes a shopper receives depends not only on his search effort, but also on the mass of active firms:

\[
N_H(c_H, v) = N_H G(u'(c_H)) + N_F G(u'(c_H)) \tag{38}
\]

\[
N_F(c_F, v) = N_H G(u'(c_F)) + N_F G(u'(c_F)) \tag{39}
\]

To avoid having to track the bounds of the support of the cost distribution, I assume that \( G(\phi) \in (0, 1) \) for all \( \phi > 0 \). Existence of an equilibrium does require that the density \( G'(\phi) \) declines to zero quickly enough as \( \phi \to 0 \). In practice, a log-normal distribution of costs seems to work well, and so would a log-logistic or Weibull distribution.

### 3.3.1 Balance of payments

After trading goods, firms can visit a perfectly competitive foreign exchange market in order to obtain the domestic currency valued by its workers and owners. Without capital markets, or in steady state, currency flows must exactly offset each other. The value of Foreign currency revenue gained by Home exports must equal the value of Home currency revenue paid for Home imports. After substituting equation (35) for the nominal
exchange rate, we obtain the balance of payments relationship:

\[ v = \frac{\Omega_H}{\Omega_F} \int_0^\infty \frac{u'(c_H)}{v} a(\tilde{G}_H(v\tau t), s_H N_H(c_H, v)) G'(t) \, dt \cdot \frac{N_F G(\frac{v}{\tau} u'(c_F))}{N_H G'(\frac{u'(c_H)}{\tau})} \]

(40)

3.3.2 Open-economy equilibrium

**Definition 2.** An open-economy equilibrium consists of seven endogenous sequences \( \{\Omega_{Ht}, \Omega_{Ft}, c_{Ht}, c_{Ft}, s_{Ht}, s_{Ft}, v_t\}_{t=0}^\infty \) which solve the following equations in each period \( t \):

I. Equation (18) for Home. In steady state, it links \( \Omega_H \) and \( c_H \). With uncertainty, it links \( \Omega_{Ht} \) to expectations about period \( t + 1 \).

II. Equation (18) for Foreign. In steady state, it links \( \Omega_F \) and \( c_F \). With uncertainty, it defines \( \Omega_{Ft} \) to expectations about period \( t + 1 \).

III. Equation (25) for Home for each \( t \), replacing \( \tilde{G}(\cdot) \) by (36) and \( N(c) \) by (38). It links \( \Omega_H, c_H, s_H, \) and \( v \).

IV. Equation (25) for Foreign for each \( t \), replacing \( \tilde{G}(\cdot) \) by (37) and \( N(c) \) by (39). It links \( \Omega_F, c_F, s_F, \) and \( v \).

V. Equation (30) for Home for each \( t \), replacing \( \tilde{G}(\cdot) \) and \( N(c) \) as for (III.). It links \( \Omega_H, c_H, s_H, \) and \( v \).

VI. Equation (30) for Foreign for each \( t \), replacing \( \tilde{G}(\cdot) \) and \( N(c) \) as for (IV.). It links \( \Omega_F, c_F, s_F, \) and \( v \).

VII. Equation (40), which links all seven endogenous variables for each \( t \).

A numerical solution to this system involves six numerical integrals, which may have to be evaluated many times as a solution is approached. This procedure can be sped up dramatically by evaluating each integral on a grid of endogenous variables and interpolating the equation on this grid. Because of the curse of dimensionality, the number of endogenous variables involved in each equation should be as small as possible.

For example, steady state solutions can be found as follows. Firstly, replace all instances of \( \Omega_H \) and \( \Omega_F \) using equations (I.) and (II.). Secondly note that while equation (VII.) still involves five endogenous variables, it can be split into two parts, the first involving the home endogenous variables plus \( v \), the second one involving the foreign endogenous variables.
variables plus $v$. There are six integrals left, each needing to be evaluated on a three-dimensional grid. If the grid contains $p$ points in one dimension, $6 \times p^3$ integrals have to be evaluated in order to create a manageable system of five interpolated equations in five variables. After this procedure, solutions can be found easily. The productivity parameters have to be hard-coded into the interpolation, but the money growth and search cost parameters can be introduced into the last stage, which makes their comparative statics very easy to examine.

4 Conclusion

In this paper, I have developed a monetary search model with heterogeneous firms and trade, amenable to studying the effect of monetary policy on markups, cost pass-through, entry, price dispersion, productivity, and welfare. At this point, it has little to say about labor and asset markets, but they could be modeled as needed. In a series of companion papers, I will aim to use the model to address the following questions:

1. Find additional empirical evidence of the mechanisms of the model.

2. What are the effects of monetary and real shocks (as in Head, Kumar, and Lapham, 2010) on exchange rates, markups, firm entry, and trade?

3. What is the optimal monetary policy, both in terms of long-run money growth and response to shocks? How should countries coordinate their monetary policies?

4. What are the effects of giving up monetary authority or forming a currency union? What are the challenges of transition into a currency union?

5. Calibrate the model to the Eurozone of the 1990s and 2000s. Ideally, the model would contribute to explaining the macroeconomic imbalances in the run-up to the Euro crisis.

References


A Appendix: Matching

The assumptions on the matching process $Q(k|\eta)$ sufficient for the results in this paper are:

M1. $Q(k|\eta)$ is a cumulative distribution function with support $\{1 \ldots K\}$, where $K \in \{2 \ldots \infty\}$. $q_k(\eta)$ is used to denote the probability mass function.

M2. $q_1(\eta) \in (0, 1)$ for all $\eta > 0$, and $q_1(0) = 1$.

M3. When $\eta' > \eta$, $Q(k|\eta') < Q(k|\eta)$: the distribution of quotes with higher search effort strictly first order stochastically dominates the one with lower search effort.

M4. $Q(k|\eta)$ is differentiable in $\eta$.

M5. $q'_1(\eta)$ is bounded as $s \to 0$.

M6. There exists an $\varepsilon > 0$ such that the series $\sum_{k=1}^{K} k^{2+\varepsilon} (q'_k(\eta))^2$ converges for any $\eta > 0$, and that the series limit approaches 0 as $\eta \to \infty$.

For quantitative results, I use the geometric matching process $q_k(\eta) = \eta^{k-1}(1+\eta)^{-k}$, which satisfies assumptions M1–6.\footnote{M1–M5 are easy to see. For $\varepsilon = 0.5$, the series limit in M6 has a closed-form solution, and it does converge to 0 as $s \to \infty$.} Intuitively, it can be characterized by the shopper flipping a (loaded) coin until it comes up tails, and then getting one quote for each flip. But all of the following matching processes give quantitatively similar results:

**Binary:** $q_1 = 1 - s$, $q_2 = s$, $K = 2$, used by Head and Kumar (2005). Advantage: if shoppers could choose each $q_k$ separately, they would choose to mix between $q_1$ and $q_2$ only. Disadvantage: consumption is a convex function of search effort, requiring a mixed strategy equilibrium for the choice of search effort.

**Poisson:** $q_k = e^{-\eta k^{-1}} k^{-1}(k-1)!$, $K = \infty$, suggested by Mortensen (2005) and used by Head and Lapham (2006). Very similar to the geometric, perhaps a bit less intuitive.

**Log-series:** $q_k = \frac{1}{\log(1+\eta)} \left(\frac{\eta}{k}\right)^k$, $K = \infty$, used in earlier versions of this paper. Slightly simpler math than the geometric, but less intuitive.
B  Appendix: Proofs

Proof of lemma 3.1: (Based on equation (34) in Burdett and Mortensen (1998).) We can rank
\[
\pi(p_1; \phi_1) \geq \pi(p_2; \phi_1) > \pi(p_2; \phi_2) \geq \pi(p_1; \phi_2) \tag{41}
\]
The first and last inequality follow from profit maximization. The middle inequality follows from \(\phi_1 < \phi_2\). Subtracting the fourth term from the first, and the third term from the second, and rearranging, we get
\[
\frac{p_2}{a(p_2)} \geq \frac{p_1}{a(p_1)} \tag{42}
\]
As \(F(p)\) is a continuous c.d.f. with connected support, it is strictly increasing on its support. From definition (2), it is easy to see that \(a(F)\) is a strictly decreasing function of \(F\). Therefore, \(a(F(p))\) is a strictly decreasing function of \(p\) on the support of \(F\). \(\square\)

Proof of lemma 3.2: \(p(\phi)\) solves the FOC. Regarding the boundary condition: either \(\bar{p}\) is binding for some sellers, in which case the marginal seller will charge \(\bar{p}\) and make zero profits. Or the least efficient seller is still able to compete, but must charge the highest price by the ranking condition; then, however, no buyer with another choice will buy from this seller. Therefore, a buyer who is willing to buy from this seller will accept any price up to \(\bar{p}\), and the least efficient seller will charge \(\bar{p}\). \(\square\)

B.1  Derivation: market equilibrium

Throughout this section, I ignore the number of firms and let \(s\) stand for the more proper \(sN\).

Consider the case where all firms have the same cost \(\phi\). Inverting equation (10), we can solve for the price a firm with rank \(F \in [0, 1]\) charges:
\[
p(F, S) = \left[ \frac{1}{\phi w} - \left( \frac{1}{\phi w} - \frac{1}{\bar{p}} \right) \frac{a(1, S)}{a(F, S)} \right]^{-1}, \tag{43}
\]
when \(S\) is the aggregate search effort in the economy. Then, a household that searches with effort \(s\), can expect the following consumption:
\[
c(s, S) = \int_0^1 \frac{a(F, s)}{p(F, S)} dF \tag{44}
\]
Substituting (43), imposing symmetry $s = S$, and substituting (19) for the wage and (7) for the reservation price, we obtain:

\[
c = - \int_0^1 \left( \frac{\Omega}{\phi} a(F, s) - \left( \frac{\Omega}{\phi} - \frac{\Omega}{u'(c)} \right) a(1, s) \right) dz = \Omega \left( \frac{1 - a(1, s)}{\phi} + \frac{a(1, s)}{u'(c)} \right),
\]

using $\int_0^1 a(F, s) dF = 1$. The result, equation (21), follows.

When firms have heterogeneous costs, we use the price function (17) in (44). We obtain, using symmetry $s = S$:

\[
c = \int_{\bar{p}/w}^{\bar{p}/w} \frac{a(\bar{G}(\phi), s)}{\bar{p}(\phi)} \bar{G}'(\phi) d\phi
\]

\[
= \int_{\phi}^{\bar{p}/w} \left[ \frac{a(1, s)}{\bar{p}} - \int_\phi^{\bar{p}/w} \frac{a_1(\bar{G}(t), s) w}{\bar{G}'(t)} dt \right] \bar{G}'(\phi) d\phi
\]

\[
= \frac{a(1, s)}{\bar{p}} - \int_\phi^{\bar{p}/w} \left[ \int_\phi^t \bar{G}'(\phi) d\phi \right] \frac{a_1(\bar{G}(t), s) w}{\bar{G}'(t)} dt
\]

\[
= \frac{a(1, s)}{\bar{p}} - \int_\phi^{\bar{p}/w} \frac{a_1(\bar{G}(t), s) w}{\bar{G}'(t)} \bar{G}(t) \bar{G}'(t) dt.
\]

Substituting $\bar{p}$ and $w$, equation (25) follows.

### B.2 Derivation: optimal search

Return to equation (44). The condition for optimal search is $u'(c) \frac{\partial c}{\partial s} = \mu$, and it yields a symmetric equilibrium when households are identical in their money holdings $M$ and their search cost $\mu$, as long as $u(c(s, S))$ is strictly concave in $s$ for any $S$\(^{10}\). Then, assuming all firms have the same cost $\phi$:

\[
\frac{\partial c}{\partial s} \bigg|_{s=S} = \int_0^1 \frac{a_2(F, s)}{p(F, s)} dF
\]

\[
= \int_0^1 \frac{a_2(F, s)}{w \phi} dF - \left( \frac{1}{\phi w} - \frac{1}{\bar{p}} \right) a(1, s) \int_0^1 \frac{a_2(F, s)}{a(F, s)} dF
\]

\[
= - \left( \frac{1}{\phi w} - \frac{1}{\bar{p}} \right) a(1, s) h(1, s)
\]

\(^{10}\) I do not have a formal analysis of this condition, but I have verified it for the geometric matching process used in this paper.
using the definition of \( h(F, s) \) (27), and the fact that \( \int_0^1 a(F, s) dF = 1 \) and therefore \( \int_0^1 a_2(F, s) dF = 0 \). Substituting \( \bar{p} \) and \( w \), we obtain:

\[
\mu = u'(c) \left( \Omega \frac{\phi}{w} - \Omega \frac{u'(c)}{u'(c)} \right) a(1, s) h(1, s).
\]

Equation (28) follows, once we account for the number of firms which is assumed to multiply \( s \). Equation (29) is obtained by dividing equation (28) by equation (21) in order to eliminate \( \Omega \), which expresses the optimal search decision in a purely contemporaneous equation.

When firms have heterogeneous costs, we again use the price function (17) in (44). We obtain, using symmetry \( s = S \):

\[
\frac{\partial c}{\partial s} \bigg|_{s=S} = \int_\phi^{\bar{p}/w} a_2(\tilde{G}(\phi), s) \tilde{G}'(\phi) d\phi
\]

\[
= \int_\phi^{\bar{p}/w} \left[ a(1, s) \frac{\tilde{G}'(t)}{\tilde{G}(t)} d\phi - \int_\phi^{\bar{p}/w} \frac{a_1(\tilde{G}(t), s)}{w t} \tilde{G}'(t) dt \right] \frac{a_2(\tilde{G}(\phi), s)}{a(\tilde{G}(\phi), s)} \tilde{G}'(\phi) d\phi
\]

\[
= \frac{a(1, s) h(1, s)}{\bar{p}} - \int_\phi^{\bar{p}/w} \frac{a_1(\tilde{G}(t), s)}{w t} h(\tilde{G}(\phi), s) \tilde{G}'(t) dt.
\]

Substituting \( \bar{p} \) and \( w \), we obtain:

\[
\mu = u'(c) \left( \Omega a(1, s) h(1, s) - \int_\phi^{u'(c)} \Omega a_1(\tilde{G}(t), s) h(\tilde{G}(\phi), s) \tilde{G}'(t) dt \right).
\]

Equation (30) follows.

**B.3 Equilibrium**

**Lemma B.1.** The functions \( a(F, s) = \sum_{k=1}^K q_k(s) k(1 - F)^{k-1} \) and \( h(F, s) = \int_0^F \frac{a_2(z, s)}{a(z, s)} dz \) have the following properties:

(i) \( a(F, s) \) is strictly decreasing in \( F \) for all \( s > 0 \), and \( a(F, 0) \equiv 1 \).

(ii) \( h(1, s) < 0 \) for \( s > 0 \), and \( h(1, 0) = 0 \).
(iii) \( a(1, s)h(1, s) \) is zero for \( s = 0 \), is negative for \( s > 0 \), and converges to 0 from below as \( s \to \infty \).

Proof. For (i), note that each component of the sum is strictly decreasing in \( F \), and the coefficients \( q_k(s)k \) are positive. By assumption M2, \( q_1(0) = 1 \), which implies that \( q_k(0) = 0 \) for all \( k \geq 2 \), therefore \( a(F, 0) \equiv 1 \). For (ii), \( a(F, 0) \equiv 1 \) implies that

\[
h(1, 0) = \int_0^1 a_2(z, 0) \, dz = \frac{\partial}{\partial s} \left[ \int_0^1 a(z, s) \, dz \right]_{s=0} = \frac{\partial}{\partial s} (1) = 0.
\]

And by assumption M3, the partial derivative \( a_2(F, s) \) is positive for low \( F \) and negative for high \( F \) (indeed, \( \int_0^1 a_2(z, 0) \, dz = 0 \)). But (i) implies that dividing by \( a(F, s) \) in the integral in \( h(1, s) \), more weight is put on high values of \( F \) and therefore on the negative parts of \( a_2(F, s) \). Assertion (ii) follows.

Concerning (iii), \( a(1, s)h(1, s) \) is zero for \( s = 0 \), and negative for any \( s \) by (ii) and by the fact that \( a(1, s) = q_1(s) \) must be strictly positive for any \( s \geq 0 \). Then, for \( \varepsilon > 0 \),

\[
(a(1, s)h(1, s))^2 = \left( \int_0^1 \frac{a(1, s)}{a(z, s)} a_2(z, s) \, dz \right)^2 
\leq \int_0^1 \left( \frac{a(1, s)}{a(z, s)} \right)^2 \, dz \cdot \int_0^1 (a_2(z, s))^2 \, dz 
\leq \int_0^1 (a_2(z, s))^2 \, dz 
= \int_0^1 \left( \sum_{k=1}^K q'_k(s)k(1-z)^{k-1} \right)^2 \, dz 
\leq \int_0^1 \left( \sum_{k=1}^K k^{2+\varepsilon}(q'_k(s))^2 \cdot \sum_{k=1}^K k^{-\varepsilon}(1-z)^{2k-2} \right) \, dz 
= \sum_{k=1}^K k^{2+\varepsilon}(q'_k(s))^2 \cdot \sum_{k=1}^K \frac{k^{-\varepsilon}}{2k-1}.
\]

The first and third inequalities are applications of the Cauchy-Schwarz inequality. The second follows from fact (i), which implies that \( a(1, s) \leq a(F, s) \) for all \( F \in [0, 1] \). What
remains is a term that converges to zero as $s \to \infty$ by assumption M6. The proof suggests that assumption M6 could be weakened if $K < \infty$.

**Full proof of theorem 3.1:** Consider equation (21). If $\Omega < u'(c)c$, as assumed (E3), it has a solution relation in $(s, c)$-space with $s > 0$ and $c > 0$. And $c \to 0$ implies $s \to 0$, as $a(1, s) = q_1(s) < 1$ with equality if and only if $s = 0$. Re-write the equation as:

$$c = \Omega \left( \frac{1 - a(1, s)}{\phi} - \frac{a(1, s)}{u'(c)} \right).$$

(45)

By assumption E2, $u'(c)$ is increasing if $c \to 0$, and more quickly than $c$ decreases itself. The term $\frac{a(1, s)}{u'(c)}$ is dominated and can be ignored when analyzing the equation in the neighborhood of $(0, 0)$. What is left is the relationship $c \approx \frac{\Omega}{\phi} (1 - a(1, s))$, which implies the implicit derivative $\frac{dc}{ds}_{s \to 0, c \to 0} = \frac{\Omega}{\phi} (-q_1'(s))$. But the derivative $-q_1'(s)$ is bounded as $s \to 0$ (by assumption M5 on the matching process). On the other hand, consider the equation (28). In the neighborhood of $(0, 0)$, its implicit derivative is infinite, i.e. it approaches the $c$-axis.

Now, let $c^*$ be the efficient level of consumption given by $(u')^{-1}(\phi)$. Equation (21) must approach $c \to c^*$, whether for $s \to 0$ or $s \to \infty$. However, the graph of equation (28) achieves a maximum $\hat{c} < c^*$ in $(s, c)$-space: by lemma B.1(iii), $-a(1, s)h(1, s)$ is positive and inverse u-shaped, and by assumption E2, the graph approaches the corners $(s \to 0, c \to 0)$ and $(s \to \infty, c \to 0)$. Consequently, the graphs of equation (21) and (28) must cross exactly once.

The steady-state analysis is slightly easier: use equation (22) to divide out $\frac{u'(c)}{\phi}$ from equation (29). The result is an equation that defines a negative relationship between $s$ and $c$:

$$\mu = (u'(c)c - \Omega) \frac{-a(1, s)h(1, s)}{1 - a(1, s)}$$

$$\Rightarrow \quad \frac{\gamma}{\gamma - \beta} \mu = u'(c)c \frac{-a(1, s)h(1, s)}{1 - a(1, s)}. \quad (46)$$

(The same as equation (48).) The graph of this equation is strictly decreasing, meets the $c$-axis, and either converges to or meets the $s$-axis. As the graph of (22) is upward sloping and approaches $c \to 0$ as $s \to 0$ and $c \to c^*$ as $s \to \infty$, the two graphs must cross exactly once. \qed
B.4 Comparative statics and the effects of inflation

Lemma B.2. Assume E1 and E2 from theorem 3.1. An increase in the production cost $\phi$ leads to a fall in consumption and a rise in search effort. An increase in the disutility of search, $\mu$, leads to a fall in both consumption and search effort. With free entry, an increase in the cost of firm entry, $k_E$, decreases consumption and has an ambiguous effect on search effort.

Proof. First, let the number of firms be fixed. Consider the version of the model where all firms have the same cost $\phi$, that is, equations (22), and (29). The search cost $\mu$ only affects the second equation, while $\phi$ affects both. But $\phi$ enters each equation as $u'(c)/\phi$, which we can therefore divide out of the second one. We obtain two equations in $(s, c)$-space:

$$\frac{\gamma}{\beta} = \frac{u'(c)}{\phi} (1 - a(1, sN)) + a(1, sN)$$

(47)

$$\mu = Nu'(c)c \left(1 - \frac{\beta}{\gamma}\right) \frac{-a(1, sN)h(1, sN)}{1 - a(1, sN)}$$

(48)

Equation (47) is identical to equation (22), which we already know slopes up in $(s, c)$-space. Furthermore, the curve shifts right/down if $\phi$ increases. Equation (48) is new but easy to understand. The term $\frac{-a(1, sN)h(1, sN)}{1 - a(1, sN)}$ is strictly decreasing in $s$,\(^{11}\) and if we assume that $u'(c)c$ is strictly decreasing in $c$, then the equation describes a downward-sloping curve in $(s, c)$-space. This curve shifts left/down if $\mu$ increases, towards the origin. Consequently, the effect of an increase in the production cost $\phi$ is a fall in consumption and a rise in search effort. On the other hand, an increase in the disutility of search, $\mu$, leads to a fall in both consumption and search effort.\(^{12}\)

If there is free entry, dividing equation (29) by equation (31) yields a curve in $(sN, c)$-space that takes the role of equation (29) in the previous discussion. Equation (21) and the new curve depend only on $sN$, not on $s$ or $N$ directly, and the new curve has the same shape as (29). So an increase in either $\mu$ or $k_E$ implies lower consumption and lower levels of $sN$. The zero-profits condition (31) implies that $N$ increases if $sN$ decreases, so search effort unambiguously falls. On the other hand, $N$ decreases in $k_E$ according to zero-profits, so the net effect on search effort is ambiguous in principle. For the parameters considered in this paper, the effect reducing search effort seems to dominate.\(^{13}\)

\(^{11}\) I have no proof yet, but I have verified it for all the cases and parameter values considered in this paper.

\(^{12}\) If $u'(c)c$ was constant, however, the curve (48) would be vertical, and if $u'(c)c$ was increasing in $c$, it would be downward sloping. In that case, an increase in $\phi$ would also reduce both consumption and search effort, provided equilibria exist for both the old and the new $\phi$.
Lemma B.3. In a steady state of a SMSE, the Friedman rule $\gamma = \beta$ implies $c = 0$, $\gamma > \beta$ implies $c > 0$ and $\gamma \to \infty$ implies $c \to 0$. Therefore, the consumption-maximizing rate of money growth satisfies $\gamma > \beta$.

Proof. Use the market equilibrium equation (22) and the contemporaneous optimal search equation (29) (or (26) and (30) with heterogeneous costs). The market equilibrium curve slopes upwards, crosses the optimal search curve exactly once, and shifts right as money growth $\gamma$ increases: faster money growth reduces the value of money, which creates market power. Constant consumption would therefore require ever-increasing search effort. As $\gamma \to \beta$, it becomes $\Gamma$-shaped and converges to the $c$-axis on the left and the $c = c^*$-line on top (in case of common costs, with $u'(c^*) = \phi$); as $\gamma \to \infty$, it converges to the $s$-axis. The graph of the contemporaneous optimal search equation (29), on the other hand, has an inverted u-shape in $(s,c)$-space, and does not depend on $\gamma$ (or any other expectations), so it contains the set of possible equilibria. It approaches $c \to 0$ both for $s \to 0$ and $s \to \infty$, because of assumptions E2, M2, and M3, and achieves a maximum $\hat{c}$ for $s > 0$. As a result, increasing money growth traces out the optimal search curve, and achieves the maximum $\hat{c}$ for $\gamma > \beta$. Higher money growth also reduces profits, which implies exit of firms, and vice versa. This tends to dampen, but not reverse, the effects of money growth on consumption and search.

Lemma B.4. Let $\mu_2 > \mu_1$.

(i) Say that the graph of the contemporaneous optimal search equation (29) achieves the maximum $\hat{c}$ for $\hat{s}$. Then $\hat{c}_2 < \hat{c}_1$ and $\hat{s}_2 > \hat{s}_1$, that is, the maximum point shifts down and to the right in $(s,c)$-space.

(ii) The consumption-maximizing rate of money growth is higher for $\mu_2$ than for $\mu_1$.

Proof. For (i), consider equation (29):

$$\mu = N u'(c) c \frac{-a(1,sN)h(1,sN) \left( \frac{u'(c)}{\phi} - 1 \right)}{u'(c) \phi (1 - a(1,sN)) + a(1,sN)}$$

Conjecture that higher $\mu$ implies lower $c$ due to the $u'(c)c$-term. Thus, $\frac{u'(c)}{\phi}$ is lower, too, so $s$ must rise a bit to make up for it. As a result, the entire curve shifts down and just a bit right. For (ii), note that the market equilibrium curve (22) is upward sloping in $(s,c)$-space. Therefore, if money growth was optimal for $\mu_1$, it must cross the new curve (29) to
the left of both the new and the old \( \hat{s} \), and higher money growth is necessary to achieve \( (\hat{s}_2, \hat{c}_2) \).

B.5 Derivation: profits, productivity, markups

Define productivity \( \varphi \) to be consumption divided by labor input. When all firms have the same marginal cost \( \phi \), it is easy to check that \( \varphi = 1/\phi \). Profits are given by (31).

When firms have heterogeneous costs, firm profits are given by (9). Using the policy function \( p(\phi) \) of (17), and defining \( \Gamma(\phi) = \int_0^\phi \tilde{G}(t) \, dt \) (the conditional mean below \( \phi \)), the aggregate profit level is:

\[
\Pi = \int_0^{u'(c)} \pi(\phi) \tilde{G}'(t) \, d\phi \\
= \int_0^{u'(c)} \left( 1 - \frac{\phi}{\Omega p(\phi)} \right) a(\tilde{G}(\phi), sN(c)) \tilde{G}'(\phi) \, d\phi \\
= 1 - \int_0^{u'(c)} \left( a(1, sN(c)) - \int_0^{u'(c)} \frac{\phi}{u'(c)} a_1(\tilde{G}(\phi), sN(c)) \tilde{G}(\phi) \, d\phi \right) \tilde{G}'(\phi) \, d\phi \\
= 1 - a(1, sN(c)) \frac{\Gamma(u'(c))}{u'(c)} - \int_0^{u'(c)} \frac{\Gamma(t)}{t} \left( -a_1(\tilde{G}(t), sN(c)) \right) \tilde{G}'(t) \, dt. 
\]

(49)

A firm with cost \( \phi \) and constant-money price \( p \) will sell \( 1/p \) units of output per consumer, and will therefore have a labor use of \( L(\phi) = \frac{\phi}{p} a(F(p), sN(c)) \). Integrating over all firms, it is easy to see the similarity with the previous calculation for profits. Indeed, each firms’ profits can be expressed as revenue—wage*labour, and since total revenue is 1 (in constant-money terms), total labor \( L \) satisfies: \( \Pi + wL = 1 \), or:

\[
L = \Omega \left( 1 - \Pi \right), 
\]

(50)

with \( \Pi \) as above or as in (32).

Since we defined productivity \( \varphi = c/L \), we have the full expression (reintroducing \( N(c) \)):

\[
\varphi(s, c) = \frac{a(1, sN(c)) \frac{\Gamma(u'(c))}{u'(c)} + \int_0^{u'(c)} \frac{\Gamma(t)}{t} \left( -a_1(\tilde{G}(t), sN(c)) \right) \tilde{G}'(t) \, dt}{a(1, sN(c)) + \int_0^{u'(c)} \frac{u'(c)}{t} \left( -a_1(\tilde{G}(t), sN(c)) \right) \tilde{G}(t) \tilde{G}'(t) \, dt}, 
\]

(51)

which is not directly dependent on expectations, hence easy to calculate given an equilibrium solution \( (s, c) \).

33
If we define a firm’s markup as its Lerner index, \( \frac{p - \phi w}{p} \), then the average markup is equal to the share of money (or output) that is retained as profits. There are two ways to see this. First, define the aggregate price level to be \( P = M/c \). Nominal costs of production are \( Mw \frac{c}{L} \) (recall that \( w \) is the constant-money wage, so \( Mw \) is the nominal wage, and \( c/L \) is productivity), and we can define the aggregate markup as \( \frac{P - Mw\frac{c}{L}}{P} = 1 - wL = \Pi \).

Alternatively, we can define each firm’s markup as \( \frac{p(\phi) - w\phi}{p(\phi)} \). We can integrate across firms, weighting by the sales of each firm:

\[
\text{markup} = \int_{\phi}^{\phi'} \left( 1 - \frac{w\phi}{p(\phi)} \right) a(\tilde{G}(\phi), s) \tilde{G}'(\phi) d\phi
\]

Accordingly, we can take total constant-money profits (i.e. the profit share of the economy) as a measure of market power, which is easier to calculate than if the markup were to be defined as \( \frac{p}{\phi w} \).

### Appendix: Illustrations

For all calculations, the representative utility function is:

\[
U(\{s_t, c_t\}_t=0^\infty) = (1 - \beta) \sum_{t=0}^\infty \beta^t \left[ \frac{c_t^{1-\sigma} - 1}{1 - \sigma} - \mu s_t - L(s_t, c_t) \right].
\]

The matching process is \( q_k(\eta) = \eta^{k-1} (1 + \eta)^{-\eta} \), which yields \( \text{mean}\{k\} = 1 + \eta \) and \( a(F, \eta) = \frac{1 + \eta}{(1 + \eta F)^2} \).

Aggregate work effort \( L(s_t, c_t) \) is defined in (49) and (50). The disutility of work effort is normalized to 1. Steady state is equivalent to a single representative period.
Figure 1: **General equilibrium in the minimal model: autarky, homogeneous firms, no entry.** The parameters are $\phi = 1$, $\beta = 0.98$, $\sigma = 2$, and $\mu = 0.008$. The continuous lines represent market equilibrium, and the contours are such that steady-state nominal interest rates would be $\{2\%, 4\%, 8\%, 16\%, 36\%\}$ (left to right, light to dark). The dashed lines represent the contemporaneous optimal search equation.

(a) Continuous: equation (22) (steady state). Dashed: equation (29).
(b) As panel (a), but in levels instead of in logs.
(c) As panel (a), except that future consumption is expected to be $e^{0.07}$. 
Figure 2: **Effects of steady-state money growth.** The parameters are $\beta = 0.98$, $\sigma = 2$, and $\mu = 0.008$ for all panels; $\phi = 1$ for panels (a) and (b); $G(\phi) \sim \logN(1, \frac{1}{6})$ for panels (c) and (d); $k_E = 0.2$ for panels (b) and (d).

(a) Autarky, homogeneous firms, no entry.

(b) Autarky, homogeneous firms, free entry.

(c) Autarky, heterogeneous firms, no entry.

(d) Autarky, heterogeneous firms, free entry.