Frictional Asset Markets and the Liquidity Channel of Monetary Policy*

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ABSTRACT

How do central bank purchases of illiquid assets affect asset prices and the real economy? To answer this question, I construct a model with heterogeneous households – some households need money more urgently than others and thus hold more of it. Households (and the government) can trade in frictional asset markets. I find that open market purchases are fundamentally different from helicopter drops: asset purchases stimulate private demand for consumption goods at the expense of demand for assets, while helicopter drops do the reverse. When assets are already scarce, further purchases crowd out the private flow of funds and can cause high real yields and disinflation – a liquidity trap.

JEL Codes: E31, E40, E50, G12

Keywords: monetary theory, monetary policy, financial frictions, indirect asset liquidity, liquidity trap, quantitative easing

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1 Introduction

Monetary policies all over the world tend to be implemented via open-market operations in financial markets. Yet, most of monetary theory has focused on 'helicopter drops' where money is given directly to households – or, if open-market operations were considered, these took place in frictionless markets with instantaneous effects and no distributional consequences. But no real-world market is perfectly frictionless, and for several notable recent examples of asset purchases, such as the quantitative easing rounds in Japan, the United States, and the Eurozone, buying up illiquid assets was the point. So in order to obtain a more robust understanding of the effects of monetary policies, I construct a model where: (i) money can be introduced either via helicopter drops or via open-market purchases; (ii) such purchases take place in frictional markets for illiquid assets; and (iii) due to market frictions and household heterogeneity, money takes time to percolate through the economy, resulting in both short-run and long-run real effects.

Specifically, I model households as transitioning between 'saver' and 'spender' states at random times; only spender households can purchase goods with money. Bonds are illiquid in the sense that they cannot be used as media of exchange, and trade in financial markets is subject to frictions. Still, bonds have a positive yield while money does not; thus, savers value bonds more than money and spenders do the opposite. If they could, they would meet up and trade until spenders hold all the money and savers hold all the bonds. But due to the trading frictions in the asset market, this cannot be done quickly enough.

The main result of the paper is therefore that the distribution of real balances matters, and not just in the short run after an intervention. Since money held by savers is idle and does not get spent on goods (which is how the prices of goods are determined), the overall *level* of real balances is affected by their *distribution*. This has consequences. For one, money injections are not neutral in the short run; for another, their effects depend on *how* the money is injected. On average, spenders hold more money than savers (as they should); thus, a helicopter drop that gives the same amount of cash to everybody shifts purchasing power away from spenders and towards savers. The result is a drop in production (due to lower demand) and a rise in asset prices (due to higher demand). This is exactly the opposite of the short-run effects of an open-market purchase in the frictional asset market: since households with a spending opportunity are the natural sellers of assets, open-market purchases make it easier for them to obtain the cash that they want. Thus, they stimulate the demand for goods at the expense of the demand for assets (net of the purchases themselves) along the transition to the new steady state.

Thus, while helicopter drops and open-market purchases are both "money injections" that increase the long-run prices of goods and assets, they are fundamentally different in the short term. In the long term, their effects point in the same direction, and their magnitude depends only on the ultimate ratio of bonds to money achieved by the intervention. Because the bonds perform a useful service in this economy – they help direct money into the hands of those who need it the most – lowering their supply reduces output.[1]

While a purchase of illiquid bonds generally has the expected effects – higher bond prices due to scarcity, and at least temporary inflation due to more money in circulation – there is an important exception. If bonds are so scarce that their marginal buyer is rationed (a liquidity trap), and if fiscal policies are fixed in real terms, then the purchase may result in *higher real bond yields and lower inflation* in the long run.[2] The reason for this surprising result is that the velocity of money is endogenous and an upward-sloping function of the bond supply inside the liquidity trap. Removing bonds from the market results in potential bond buyers holding on to their money for longer, a textbook increase in "money demand" that defeats the intent of the expansion of money supply. In contrast to almost all formal models of liquidity traps, here interest rates are positive throughout since the bonds are long-term.

My model is a hybrid of a New Monetarist model in the tradition of Lagos and Wright (2005) (surveyed by Nosal and Rocheteau, 2011, and Lagos, Rocheteau, and Wright, 2017) and a model of frictional asset markets in the tradition of Duffie, Gârleanu, and Pedersen (2005) (DGP, henceforth). The most closely related New Monetarist model is the one of Rocheteau, Weill, and Wong (2015) (RWW, henceforth); from it, I adopt the monetary environment and the structure of goods and labor markets. I introduce two innovations: first, households are heterogeneous in how soon they expect to need money, and second, there are financial assets in addition to money which can only be traded in frictional asset markets. The structure of these asset markets is adapted from DGP. Compared to that paper, there are three innovations: first, assets are traded for money rather than transferable utility; second, random liquidity needs provide a microfoundation of DGP's exogenous asset valuation shocks (as originally suggested by DGP); and third, all assets are perfectly divisible.[3]

The literature which applies insights from monetary theory to asset pricing is extensive and falls into two broad groups. In the first, financial assets directly compete with money as the medium of exchange (Geromichalos, Licari, and Suárez-Lledó, 2007; Lagos, 2010, 2011; Rocheteau and Wright, 2012); hence, this can be called the "direct liquidity" approach. Here,

[1] If the bonds are nominal, then the money expansion implicitly decreases their *real* supply, as was pointed out by Williamson (2012). If the bonds are real, then in the long run a helicopter drop has no effect (because money is classically neutral) but an open-market purchase does (because there are fewer bonds), as was pointed out by Rocheteau, Wright, and Xiao (2018).

[2] Similar results appear in Williamson (2012) and Andolfatto and Williamson (2015), but only when lower government debt results in a lower need for seigniorage revenue. Here, the result can obtain even for *constant* real seigniorage, because real balances increase.

[3] Naturally, adding these ingredients entails compromise in other areas. Here, I restrict attention to quasilinear utility, in contrast to RWW who use more general preferences; see the discussion on page 4.3. Compared to DGP, asset trade is simplified in that there are no dealers, and prices are competitive rather than bargained; but see the extension in Appendix A.3. Geromichalos and Herrenbrueck (2016a) also take up DGP's suggestion to microfound the asset valuation shocks as heterogeneous liquidity needs, but only focus on the long run.

however, I study the pricing of an asset that cannot be used in exchange but has endogenous liquidity properties because it can be traded for money in an asset market. Thus, it inherits 'moneyness' from the fact that people who anticipate needing money in the future know that they can *liquidate* their other asset when the need arises. This "indirect liquidity" approach unites the second group of papers, including Geromichalos and Herrenbrueck (2016a), Lagos and Zhang (2015), Berentsen, Huber, and Marchesiani (2014), Mattesini and Nosal (2016), Herrenbrueck and Geromichalos (2017), and Huber and Kim (2017) (with Berentsen, Camera, and Waller, 2007, and Berentsen and Waller, 2011, as precursors).[4]

As is well-known, open-market operations are neutral when conducted by swapping liquid money for illiquid bonds in frictionless markets and fiscal implications are sterilized (Sargent and Smith, 1987). But they can have real effects on the economy when any of these conditions is not satisfied; indeed, in this paper, I study interventions in frictional markets for imperfectly liquid bonds throughout, and allow fiscal implications for a sub-section of my analysis. Previous New Monetarist work has also studied open-market operations in environments where one or two of these conditions was relaxed, but not all three. Williamson (2012) and Rocheteau, Wright, and Xiao (2018) also study purchases of partially liquid bonds but model them as happening in a frictionless market. Neither paper looks at short-run dynamics. Andolfatto and Williamson (2015) and Geromichalos and Herrenbrueck (2017) do also study short-run dynamics, and they add an extra wrinkle: they identify the yield on the partially liquid bond as the main monetary policy instrument, so that the monetary authority "sets" this yield and conducts open-market operations to achieve it in the background.

The argument that frictions in portfolio management are the source of monetary non-neutrality and make intervention effective has a long tradition in monetary theory (Baumol, 1952; Tobin, 1956), continued by Alvarez and Lippi (2009, 2013). This approach evolved into the "limited participation" literature, in which not all agents participate in asset markets, and some agents face spending or borrowing constraints (Fuerst, 1992; Alvarez, Atkeson, and Kehoe, 2002; Williamson, 2006). One closely related paper is by Alvarez and Lippi (2014), who also study the interaction of asset market frictions, portfolio heterogeneity, and monetary policy, and derive some similar results (e.g. the short-term liquidity effect of money on interest rates). There are two main differences. First, in Alvarez and Lippi (2014), assets enter the utility function (and key results are derived from the shape of that function) whereas in my paper, money has value because it is the medium of exchange (and key results are

[4] Applying models of asset market frictions to markets such as those for government bonds is sometimes challenged because these markets are considered highly liquid. However, the frictional model is valid as long as the markets are not *perfectly* liquid, which of course no real-world market is. For example, Ashcraft and Duffie (2007) documented the importance of frictions in the federal funds market, which at the time was one of the most liquid markets in existence. Vayanos and Weill (2008) and Andreasen, Christensen, Cook, and Riddell (2016) attribute yield spreads of 30-60 basis points between different classes of Treasuries to differences in liquidity. Finally, if Treasuries were exactly as liquid as cash then they could not be priced at a positive nominal yield by agents who also hold money. But they are.

derived from the properties of the asset markets). Second, Alvarez and Lippi (2014) analyze an endowment economy and focus on stationary correlations between financial variables, whereas I model a production economy and focus on the effects of a one-time intervention.

Finally, in its emphasis on household heterogeneity as a driver of dispersed portfolios and ongoing asset trade, this paper also continues a tradition going back to Bewley (1980, 1983), Scheinkman and Weiss (1986), and Kehoe, Levine, Woodford, et al. (1992) (all of which are single-asset models where the only policies considered are lump-sum interventions). A recent contribution by Cúrdia and Woodford (2011) has in common with my paper that households are heterogeneous and differ in their demand for liquidity; they model it as patience shocks that make some households want to borrow, whereas I model it as random differences in how soon households expect to need money.

The paper is organized as follows. Section 2 develops the model; Section 3 solves for and characterizes its equilibria. In Section 4, the model is used to analyze monetary neutrality, the long-run and short-run liquidity channel, and the liquidity trap. Section 5 concludes. The Appendix contains proofs and further details.

2 A model of frictional asset markets

The rest of this paper focuses on an economy with two assets – money and government bonds – and monetary policy intervention in the bond market. The working paper version of this paper (Herrenbrueck, 2014) also contains real bonds and an extension of the model to physical capital, with a segmented frictional capital market; it explores the substitutability of bonds and capital and the effect of monetary policy on investment dynamics.

2.1 Environment

Time $t = [0, \infty)$ is continuous. There are two types of agents: households and a government. Households have unit measure and are infinitely lived. The government is a single consolidated authority that can create assets, make transfers, and collect taxes.

There are four commodities in the economy. The first is a lumpy consumption good, denoted by c, which can only be consumed as a stock at certain random opportunities. It will serve as the numéraire in this economy. The second is labor effort, denoted by h, which is expended as a flow. Both of these commodities are perishable and generate utility. The other two commodities are assets, perfectly durable and divisible. The first is fiat money, denoted by M and measured in dollars; the second is a nominal perpetuity bond B, which pays a flow dividend of one dollar per unit of time. The supplies of money and bonds, M_t and B_t, are controlled by the government.

Households cycle through a Markov process between two states, called 0 and 1. In state 1,

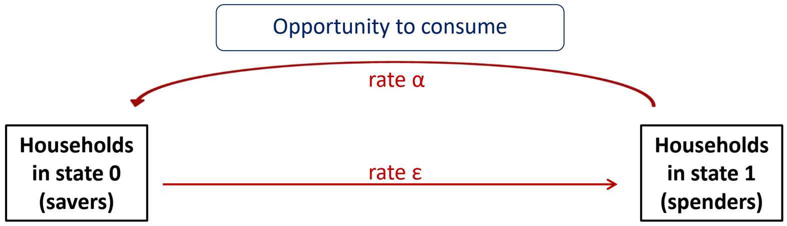


Figure 1: Households cycle through a two-state Markov process

households receive opportunities to consume the good c at Poisson arrival rate $\alpha > 0$. Immediately after such shocks, they transition back into state 0, an assumption which is made without loss of generality. While in state 0, households never receive such opportunities, but they randomly transition to state 1 at Poisson arrival rate $\varepsilon > 0$. Hence, the process represents *information*: state 1 represents a 'high likelihood to consume' (because it takes $1/\alpha$ years on average for the consumption shock to arrive), while state 0 represents a 'low likelihood to consume' (because it takes $1/\alpha + 1/\varepsilon$ years on average for both shocks to arrive). Even more simply put, one can think of the households in state 0 as *savers* and of households in state 1 as *spenders*.[5] Figure 1 provides an illustration.

Households discount time at rate $\rho > 0$. Each household owns $\bar{h} < \infty$ units of labor. By working, a household can transform labor h into the good c at a constant marginal cost of 1. Labor effort h generates flow disutility $v(h)$, and consumption of c units of the consumption good (at random time T_1) generates utility c: thus, marginal utility is constant and normalized to 1. As a result, we can represent the expected utilities of a household in state 0, $U_{0,t}$, and in state 1, $U_{1,t}$, in recursive-utility form:

$$U_{0,t} = \mathbb{E}\left\{-\int_t^{T_\varepsilon} \left[e^{-\rho(\tau-t)}\, v(h_\tau)\right] d\tau + e^{-\rho(T_\varepsilon - t)}\, U_{1,T_\varepsilon}\right\}$$
$$U_{1,t} = \mathbb{E}\left\{-\int_t^{T_\alpha} \left[e^{-\rho(\tau-t)}\, v(h_\tau)\right] d\tau + e^{-\rho(T_\alpha - t)}\, [c_{T_\alpha} + U_{0,T_\alpha}]\right\}$$

where the first expectation is over the random time T_ε (which arrives at rate ε), and the second expectation is over the random time T_α (which arrives at rate α).

The function v is strictly increasing and convex. Furthermore, assume $v'(0) = 0$ and $v'(\bar{h}) > 1$; given that the marginal utility of consumption is 1, this implies that in the full-insurance benchmark and in any equilibrium, the constraint $h \leq \bar{h}$ will never bind. Let n_i denote the measure of households in state $i = 0, 1$. In steady state, we must have $n_0 = 1 - n_1 = \alpha/(\varepsilon + \alpha)$, and to make things simpler, I maintain for the rest of the paper the assumption that these measures are in steady state at all times.

[5] Just as interesting might be an alternative model where the state process represents income shocks rather than demand shocks, similar to Scheinkman and Weiss (1986) or Wong (2016). But that is left for future work.

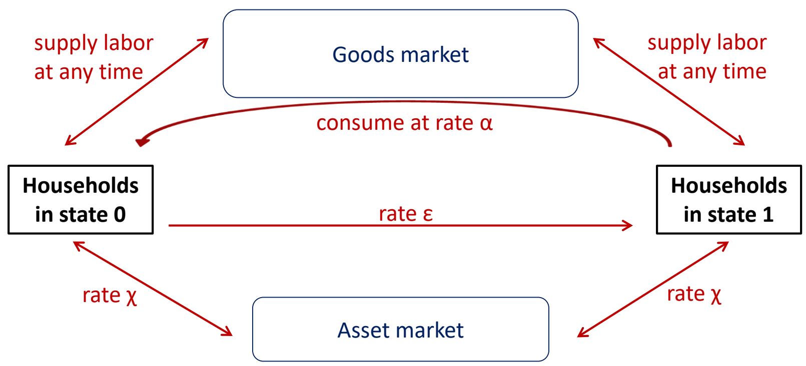


Figure 2: Illustration of the market structure sans government intervention

There are two markets, illustrated in Figure 2: a goods market and an asset market. Households are anonymous and take prices as given in all markets. In the goods market, all households use their labor to produce goods, while only those households who are hit with the α-shock demand goods. This gives rise to a quantity mismatch: at each point in time, a positive measure of workers is producing an infinitesimal quantity of goods (that is, a flow), while an infinitesimal measure of consumers wants to consume a positive quantity of goods (a stock). Because of this mismatch, and because credit is not feasible due to anonymity, a medium of exchange is necessary. For the purpose of this paper, I assume that only money can be recognized by everyone; households who own bonds know what their own bonds look like, but they cannot verify the authenticity of the bonds that a prospective buyer holds. Hence, money is the only feasible means of payment.[6] Denote the price of money in terms of output by ϕ, so that the inflation rate is $\pi \equiv -\dot{\phi}/\phi$. The rate of transformation of labor into goods is 1, thus the real wage is 1 at all times.

Access to the asset market is subject to a friction: households in either state enter the market randomly at Poisson rate $\chi > 0$. When in the market, households can offer to sell as many bonds as they own, or ask to buy as many bonds as they have money to afford, at price q. As in the goods market, pricing is competitive.[7]

The government can make nominal lump-sum transfers T to households (or collect taxes if $T < 0$). They are lump-sum in terms of applying equally to all households, but it is important to keep in mind that they are being assessed as *flows*: they affect the rate of change

[6] Nosal and Wallace (2007), Rocheteau (2009, 2011), Li and Rocheteau (2011), and Lester, Postlewaite, and Wright (2011) analyze in more depth how money emerges as the medium of exchange; Geromichalos, Jung, Lee, and Carlos (2019) do so in an environment where secondary asset markets exist as well. And in Appendix A.4, I study an extension where bonds can also be used as a means of payment, but with a probability less than one.

[7] One may ask why households can recognize the bonds owned by other households in the asset market, but not in the goods market. For the purposes of this paper, this assumption is taken as a primitive; the markets are simply different. One way to make this distinction rigorous is by introducing dealers that intermediate asset trade; these dealers are able to verify and certify the authenticity of a bond, and they are compensated for their service with a bid-ask spread. See Appendix A.3 for details.

of households' money holdings, not the holdings directly. The government also has to service its debt by paying a flow dividend of one dollar per year to the owner of a bond. The government can sell new bonds at flow rate $S \in \mathbb{R}$ in the decentralized bond market (or buy them back if S is negative).

2.2 The full-insurance benchmark

Suppose that all households pool together their labor efforts to insure themselves against the consumption shock. They maximize welfare, subject to the aggregate resource constraint:

$$\max\left\{\int_0^\infty e^{-\rho t}\Big(-v(h_t)+\alpha n_1 c_t\Big)dt\right\} \quad \text{subject to:} \quad \alpha n_1 c_t = h_t \quad \text{and} \quad h_t \leq \bar{h},$$

where c_t is the stock of consumption goods given to all consuming households and h_t is the flow of labor services each household contributes. (Since all households equally dislike working, it is optimal for all to work the same amount.) As the marginal utility of consuming and the marginal rate of transforming labor into consumption are both normalized to 1, the solution sets the marginal disutility of working to equal 1 at all times, too:

$$v'(h^{FI}) = 1 \qquad \text{and} \qquad h^{FI} = \left(\frac{1}{\varepsilon}+\frac{1}{\alpha}\right)c^{FI} \qquad \forall t \geq 0 \tag{1}$$

For interpretation, recall that the term $\alpha n_1 = (1/\varepsilon + 1/\alpha)$ is the expected length of time to elapse between two consumption opportunities.

3 Stationary monetary equilibria

In this section, I study stationary monetary equilibria where aggregate levels of real balances and the real bond supply are constant. (The definition will be extended to dynamic equilibria in Appendix A.2.) This requires that all nominal variables grow at the same rate $\gamma > -\rho$, and it implies that the inflation rate is $\pi = \gamma$. Thus, the value of money satisfies $\phi_t = \phi_0 \exp(-\gamma t)$, where ϕ_0 is determined in equilibrium.

3.1 Household's problem

For convenience, the household's problem is stated and solved in real terms: let $z_t^M \equiv \phi_t m_t$ denote real money balances and $z_t^B \equiv \phi_t b_t$ denote real bond holdings, and denote the real fiscal transfer by $\Upsilon \equiv \phi_t T_t$. Households decide on the flow of labor effort h_t, how many real balances z_t^M to accumulate, and how much to trade when accessing the goods or asset market. Recall that the real wage is 1, and the price of consumption in terms of real balances is 1 by definition. When given a random opportunity to consume, a household with z units of real balances chooses to purchase $c(z^M) \in [0, z^M]$ units of consumption. When in the asset market, a household in state i with z^M real balances and z^B real bonds chooses to sell $s_i(z^M, z^B) \in [-z^M/q, z^B]$ units of bonds, at the prevailing market price q. (If $s_i < 0$, then the household is a buyer of bonds.)

Let $W_i(z^M, z^B)$ denote the maximum attainable lifetime utility of a household in state i holding a portfolio (z^M, z^B) of real money and bonds.

The state transition times T_ε and T_α are exponentially distributed with parameters ε, α, respectively, and the arrival times of the asset market access shocks are exponentially distributed with parameter χ. Since the arrival time of the *first* of multiple Poisson shocks is exponentially distributed, too, and its parameter is the sum of the Poisson arrival rates, the value functions W_0 and W_1 solve the recursive optimization problems:

$$W_0(z^M, z^B) = \sup \int_0^\infty e^{-(\rho+\varepsilon+\chi)t} \Big[- v(h_t) + \varepsilon\, W_1(z_t^M, z_t^B) + \chi\, W_0(z_t^M + q_t s_{0,t}, z_t^B - s_{0,t}) \Big] dt \tag{2a}$$

$$W_1(z^M, z^B) = \sup \int_0^\infty e^{-(\rho+\alpha+\chi)t} \Big[- v(h_t) + \alpha \left[c_t + W_0(z_t^M - c_t, z_t^B)\right] + \chi\, W_1(z_t^M + q_t s_{1,t}, z_t^B - s_{1,t}) \Big] dt \tag{2b}$$

where both problems are with respect to left-continuous plans for $\{h_t, c_t, s_{0,t}, s_{1,t}\}$, piecewise continuously differentiable plans for $\{z_t^M, z_t^B\}$, and subject to:

$$\dot{z}_t^M = z^B + h_t - \gamma z_t^M + \Upsilon, \qquad z_0^M = z^M, \tag{3a}$$

$$\dot{z}_t^B = -\gamma z_t^B, \qquad z_0^B = z^B, \tag{3b}$$

$$-z_t^M/q_t \le s_{0,t} \le z_t^B, \qquad -z_t^M/q_t \le s_{1,t} \le z_t^B \tag{3c}$$

$$0 \le c_t \le z_t^M, \qquad 0 \le h_t \le \bar{h}. \tag{3d}$$

The effective discount factors $\exp[-(\rho+\varepsilon+\chi)t]$ and $\exp[-(\rho+\alpha+\chi)t]$ are the products of the discount factor, $\exp(-\rho t)$, with the probabilities that no transition or trading opportunity shock occurs during the time interval $[0, t)$.

Proposition 1. *Given $\Upsilon \ge 0$, Equations (2) have a unique bounded solution of functions $W_0(z^M, z^B)$ and $W_1(z^M, z^B)$. They are strictly increasing, continuously differentiable over $[0, \infty)^2$, and satisfy the Hamilton-Jacobi-Bellman equations:*

$$(\rho + \varepsilon + \chi) W_0(z^M, z^B) = \max_{h_0, s_0} \Big\{ - v(h_0) + \varepsilon\, W_1(z^M, z^B) + \chi\, W_0(z^M + q s_0, z^B - s_0) \tag{4a}$$

$$+\frac{\partial}{\partial z}W_0(z^M,z^B)\left[z^B+h_0-\gamma z^M+\Upsilon\right]+\frac{\partial}{\partial b}W_0(z^M,z^B)\left[-\gamma z^B\right]\Big\}$$

$$\textit{subject to}\quad 0\le h_0\le\bar{h}\quad\textit{and}\quad -z^M/q\le s_0\le z^B$$

$$(\rho+\alpha+\chi)W_1(z^M,z^B)=\max_{h_1,s_1,c}\Big\{-v(h_1)+\alpha\left[c+W_0(z^M-c,z^B)\right]+\chi\,W_1(z^M+qs_1,z^B-s_1)\quad\text{(4b)}$$

$$+\frac{\partial}{\partial z}W_1(z^M,z^B)\left[z^B+h_1-\gamma z^M+\Upsilon\right]+\frac{\partial}{\partial b}W_0(z^M,z^B)\left[-\gamma z^B\right]\Big\}$$

$$\textit{subject to}\quad 0\le h_1\le\bar{h},\quad -z^M/q\le s_1\le z^B,\quad\textit{and}\quad 0\le c\le z^M$$

Proof. See Appendix A.1. □

The proposition establishes the regularity of the household's problem. Solving this problem yields strong results: the value functions are linear in both arguments, labor supply decisions are smooth, and spending decisions are bang-bang. Formally, these results can be stated as follows:

Proposition 2. *(a) The value functions are linear:*

$$W_i(z,b)=W_i(0,0)+\mu_i\,z^M+\beta_i\,z^B,\tag{5}$$

so that μ and β are the costate variables associated with real units of money and bonds, respectively. Given inflation ($\pi=\gamma$) and given bond price (q), the costates satisfy the following Euler equations:

$$\rho\mu_0=-\gamma\mu_0+\varepsilon(\mu_1-\mu_0)+\chi\max\left\{\frac{\beta_0}{q}-\mu_0,\,0\right\}\tag{6a}$$

$$\rho\mu_1=-\gamma\mu_1+\alpha(1-\mu_1)+\chi\max\left\{\frac{\beta_1}{q}-\mu_1,\,0\right\}\tag{6b}$$

$$\rho\beta_0=-\gamma\beta_0+\mu_0+\varepsilon(\beta_1-\beta_0)+\chi\max\left\{q\,\mu_0-\beta_0,\,0\right\}\tag{6c}$$

$$\rho\beta_1=-\gamma\beta_1+\mu_1+\alpha(\beta_0-\beta_1)+\chi\max\left\{q\,\mu_1-\beta_1,\,0\right\}\tag{6d}$$

(b) Given the costate solutions, labor supplies satisfy:

$$v'(h_0)=\mu_0\qquad\textit{and}\qquad v'(h_1)=\mu_1\tag{7}$$

(c) Furthermore, spending decisions satisfy, for all $t\ge 0$:

$$c_t=z_t^M\qquad\textit{and}\qquad s_{i,t}=\begin{cases}-z_t^M/q & \textit{if } q<\beta_i/\mu_i\\ \in[-z_t^M/q,\,z_t^B] & \textit{if } q=\beta_i/\mu_i\\ z_t^B & \textit{otherwise}\end{cases}\tag{8}$$

(d) Finally, the costates satisfy the following inequalities:

$$\mu_0 < \mu_1 < 1, \qquad \frac{\beta_0}{\mu_0} > \frac{\beta_1}{\mu_1}, \qquad \textit{and} \qquad \frac{\beta_0}{\mu_0} > \frac{1}{\rho + \gamma} \tag{9}$$

In the special case of $q = \beta_1/\mu_1$, we additionally have:

$$q = \frac{\beta_1}{\mu_1} < \frac{1}{\rho + \gamma} \tag{10}$$

Proof. See Appendix A.1. □

Equations (6) have straightforward asset pricing interpretations. For example, the marginal flow value of real balances to households in state zero ($\rho\mu_0$) is composed as follows: first, they lose value to inflation ($-\gamma\mu_0$); second, they gain value in transition to state 1 ($\varepsilon[\mu_1 - \mu_0]$); and finally, they can be used to buy bonds at price q_t when the household has access to the asset market ($\chi\,[\beta_0/q - \mu_0]$). If bonds are too expensive, however, then the $\max$-term is zero because the household will hold on to its money rather than spend it on bonds. The other equations admit analogous interpretations. The term μ_i in the value of bonds represents the bond dividend, normalized to a flow of one dollar per year.

The fact that the value functions are linear in a household's asset holdings has two important consequences. Firstly, the choice of labor effort depends only on which state a household is in, not on its wealth or the composition thereof. As v is strictly convex, households with a high value of money work harder and accumulate real balances faster than those with a low value of money.

Secondly, linearity implies that the spending decisions in the goods or asset market are bang-bang: depending on the price, and unless they are exactly indifferent, households either spend everything or nothing. Thus, households with a consumption opportunity will spend all their money on goods if their valuation of the goods exceeds the value of holding on to their real balances: $1 > \mu_0$. And households with an opportunity for asset trade will spend all their money to buy bonds if the household's marginal rate of substitution exceeds the market price: $\beta_i/\mu_i > q$. Conversely, households will sell all their bonds if $\beta_i/\mu_i < q$.

Finally, the proposition verifies that trade will occur in the natural directions: households with an opportunity to spend money will do so, and households in state 1 (who value money more) will sell bonds to households in state 0 (who value bonds more, *relative to money*). One implication is that the asset trade terms in Equations (6b) and (6c) must equal zero in a steady-state equilibrium: e.g., households in state 0 do not value bonds for their ability to be sold, because they are planning to buy *more* bonds in case they gain access to the asset market. Figure 3 illustrates an example path of a household's money holdings, as the various transitions and trades occur.

As the last inequality in (9) shows, the reservation price of bonds for households in state 0 always exceeds the "fundamental price" $1/(\rho+\gamma)$. The difference is a "liquidity premium",

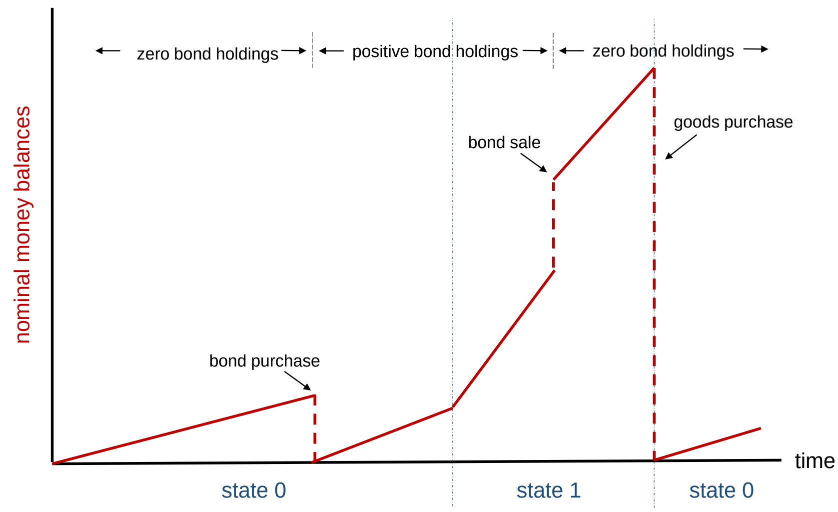


Figure 3: Sample path of a household's money holdings

because the bond helps such households store their wealth in a form other than money, but nevertheless somewhat liquid. By contrast, the reservation price of bonds for households in state 1 may be greater than the fundamental – reflecting an anticipated *future* liquidity premium – or smaller. In the latter case (inequality (10)), the market price of bonds includes an "illiquidity discount"; bond sellers would like to liquidate their bond holdings before they need the money, but with some probability will fail to do so.

3.2 Equilibrium

In equilibrium, the money market, goods market, and asset market must clear, and the government's choices must satisfy a budget constraint. In order to describe aggregate flows through the markets, let $M_i \equiv n_i z_i^M/\phi$ and $B_i \equiv n_i z_i^B/\phi$ denote the *total* stocks of money and bonds held by households in state i.

The money market clears if $M_0 + M_1 = M$, i.e., the demand for money equals its supply. The goods market clears if the inflow of money from buyers matches the outflow to working households. As households in state 1 buy goods at flow rate α, and each such household spends all of its money, the flow of real balances into the market is $\alpha\phi M_1$. In return, working households earn a real income flow of h_i, so the total flow of real balances out of the goods market is $n_0 h_0 + n_1 h_1$. The equality of these flows determines the value of money, ϕ:

$$\alpha\phi M_1 = n_0 h_0 + n_1 h_1 \tag{11}$$

The unconstrained flow of money into the bond market is χM_0, and the unconstrained inflow of bonds is $\chi B_1 + S$ (where S represents the flow of open-market sales by the government). The ratio of the unconstrained flows defines a *candidate* market price:

$$\tilde{q} \equiv \frac{\chi M_0}{\chi B_1 + S}$$

If buyers are willing to pay this candidate price and sellers are willing to receive it (that is, $\tilde{q} \in [\beta_1/\mu_1, \beta_0/\mu_0]$), then $\tilde{q}$ clears the bond market.[8] However, if this is not the case, then some traders must be rationed. Denote by $\psi_i \in [0,1]$ the probability that a household in state i gets to trade; we assume for simplicity that the government never gets rationed.[9] Naturally, $\psi_i < 1$ can only be part of an equilibrium if $q = \beta_i/\mu_i$; that is, households on the long side of the market are indifferent between trading or keeping their assets. Bond market clearing can thus be expressed as equality of the constrained flows of money and bonds:

$$\underbrace{\chi\psi_0 M_0}_{\text{inflow of money}} \quad = \quad \underbrace{(\chi\psi_1 B_1 + S)\, q}_{\text{outflow of money}}$$

with solution:

$$q = \max\left\{\frac{\beta_1}{\mu_1}, \min\left\{\tilde{q},\, \frac{\beta_0}{\mu_0}\right\}\right\}; \qquad \psi_0 = \min\left\{\frac{\beta_0}{\mu_0}\,\frac{1}{\tilde{q}},\, 1\right\}; \qquad \psi_1 = \min\left\{\tilde{q}\,\frac{\mu_1}{\beta_1},\, 1\right\} \tag{12}$$

The government must finance a flow of transfers T (or has access to taxes if $T < 0$) and dividend payments on the outstanding debt. As each bond pays a flow dividend of one dollar, the total dividend flow is B dollars. If the money supply grows at rate $\dot{M} = \gamma M$, then the government also has access to seigniorage revenue $\dot{M}$ in nominal terms (thus, $\phi\dot{M}$ in real terms). Finally, the government may have income from open-market sales of bonds, $q \cdot S$ (or must finance open-market purchases if $S < 0$). Its flow budget constraint is therefore:

$$T + B = \gamma(M_0 + M_1) + q\,S \tag{13}$$

In a steady state with a constant money-bonds ratio M/B, the bond supply must also grow at rate γ. Thus, $S = \gamma B$ in steady state.

Figure 4 illustrates the flows of money between agents in the model, and Figure 5 does the same for bonds. Equalizing inflows and outflows for the two markets and the govern-

[8] As long as the intervention is not too large – specifically, $S \in [-\chi B_1, \chi M_0/q]$ – it changes the magnitude of private asset flows but not their direction. Here, I assume that this is always the case, although it would in principle be possible for the government to purchase bonds at such a pace (i.e., at such an attractive price) that *everyone* would want to sell them for money. In that case, the distributional effects of the intervention would mostly go away (because everyone becomes a bond seller in the market, with the government on the other side). The long-term effects (of a lower bond stock) would remain the same. For the numerical experiments in this paper, I verify numerically that the interiority assumption holds throughout.

[9] This is without loss of generality, since households will be rationed only if they are indifferent to being rationed. Indifference also explains why ψ_0 and ψ_1 do not enter the household's problem: households expect a surplus from asset trade only if they expect to get served with probability one.

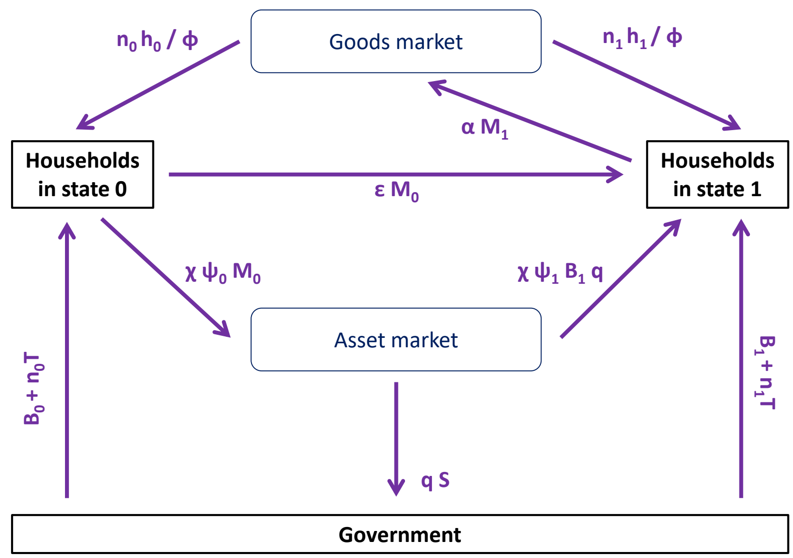


Figure 4: Flows of money between groups of agents

ment yields Equations (11), (12), and (13) (because the government does not hold an inventory of assets). What is left is to describe accumulation of assets by households. Fortunately, as explained above, all households in a given state choose identical values of labor effort, which we denote by h_0 and h_1. Accounting for the flow of assets to and from households in state 0 or state 1 is then straightforward:

$$\dot{M}_0 = B_0 + n_0(h_0/\phi + T) - \varepsilon M_0 - \chi\psi_0 M_0 \tag{14a}$$

$$\dot{M}_1 = B_1 + n_1(h_1/\phi + T) + \varepsilon M_0 + \chi\psi_0 M_0 - q\,S - \alpha M_1 \tag{14b}$$

$$\dot{B}_0 = -\varepsilon B_0 + (\alpha + \chi\psi_1)B_1 + S \tag{14c}$$

$$\dot{B}_1 = \varepsilon B_0 - (\alpha + \chi\psi_1)B_1 \tag{14d}$$

For example, the stock of money held by households in state 0 increases due to dividend income (B_0), labor income ($n_0 h_0/\phi$ when measured in dollars), and transfer income ($n_0 T$); it decreases via transition to state 1 by some households (εM_0), and expenditure on bonds in the asset market ($\chi\psi_0 M_0$). The other equations have analogous interpretations in terms of inflows and outflows.

Definition 1. A stationary monetary equilibrium is a list of constant variables $\{h_0, h_1, \mu_0, \mu_1, \beta_0, \beta_1, q, \psi_0, \psi_1\}$, variable ϕ_t declining at rate γ, and a list of variables $\{M_{0,t}, M_{1,t}, B_{0,t}, B_{1,t}, T_t, S_t\}$ growing at rate γ, which satisfy Equations (6), (7), (11), (12), (13), (14), $S_t = \gamma B_t$, $M_{0,t} + M_{1,t} = M_t$, and $B_{0,t} + B_{1,t} = B_t$ for all $t \geq 0$, as well as $\phi_0 > 0$.

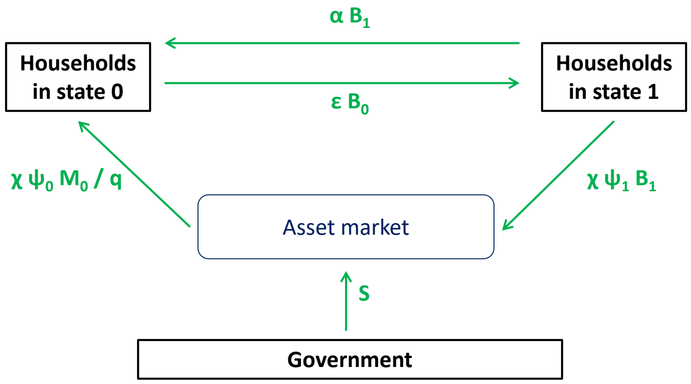


Figure 5: Flows of bonds between groups of agents

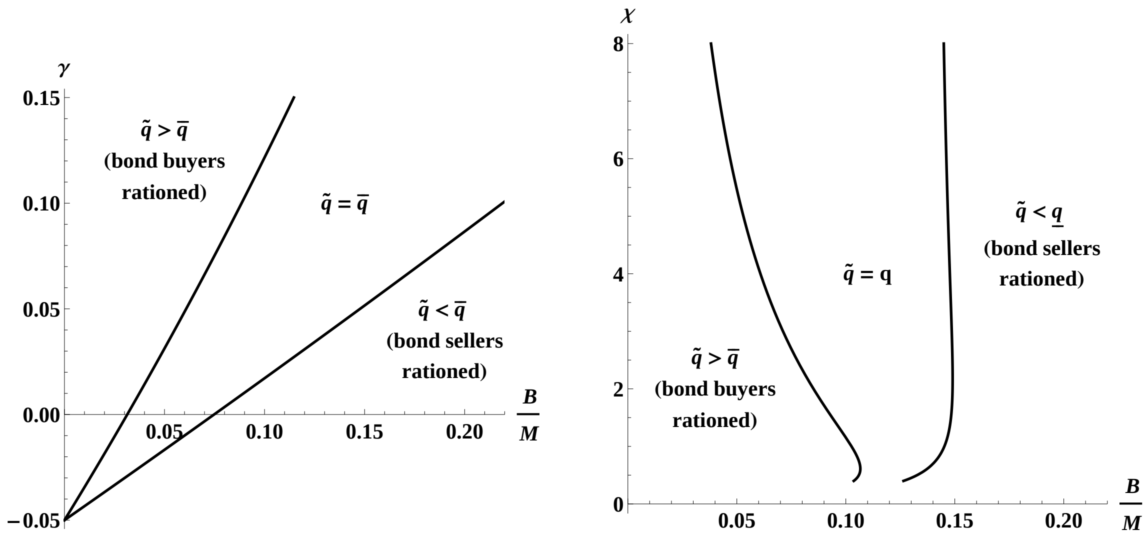
In the definition above, government transfers T are treated as an endogenous variable that adjusts to satisfy the government budget constraint, while government debt B and the rate of money growth γ are exogenous. Since choosing γ is the purview of monetary policy, and since this choice imposes a constraint on the fiscal tradeoff between T and B, this assumption is one of **monetary dominance**. Alternatively, we could assume **fiscal dominance**: T and B are exogenous and γ adjusts to satisfy the budget constraint.[10]

As usual, a "monetary" equilibrium is one in which at least *some* households *value* money ($\phi > 0$). However, this model is tractable only if a stronger condition holds: *all* households *accumulate* money in equilibrium. The reason is technical. Even if a household did not want to accumulate money, it might still value having some money in order to pay its taxes. But such a household would *decumulate* money and hit the constraint $z \geq 0$ in finite time; anticipating this, its value function would not be linear in money. Consequently, households' willingness to pay for assets would be heterogeneous and their decisions could not be aggregated in the simple form shown above. To keep things simple, I assume that taxes will never be so large as to force a household to decumulate money. As I show in Appendix A.1, in steady state this is equivalent to the following inequality:

$$\frac{B}{M} \leq \gamma\left(1 + q\frac{B}{M}\right) + \alpha\,\frac{h_0}{n_0 h_0 + n_1 h_1}\,\frac{M_1}{M} \tag{15}$$

If money is printed fast enough, then bond dividends can be paid out of seigniorage revenue, without needing to impose any taxes. (It is possible that $\gamma q > 1$; in that case, the real return on bonds is less than inflation and the government can issue bonds 'for free'.) Additionally,

[10] The literature has not yet settled on a naming convention for this important distinction. Alternative terms for monetary dominance include "active fiscal policy" and "Ricardian fiscal policy", with the counterparts for fiscal dominance being "passive fiscal policy" and "non-Ricardian fiscal policy".
Defining equilibrium under fiscal dominance is analogous; however, such an equilibrium would not be unique, because real seigniorage revenue is subject to a Laffer curve.



[a] Fixing $\chi = 4$

[b] Fixing $\gamma = .05$

Figure 6: Regions of stationary equilibria, as functions of bonds-money ratio B/M, money growth γ, and trading frequency χ. Key parameters: $\rho = .05$, $\varepsilon = 1$, $\alpha = 2$.

the fractional term equals the maximal real tax flow that *every* household can pay out of its current income. However, while intuitive, this condition involves endogenous variables. A sufficient condition that only involves exogenous variables is:

$$\frac{B}{M} \leq \min\left\{ \frac{1}{1-\gamma \underline{q}} \left(\gamma + \frac{\varepsilon\alpha}{\varepsilon+\alpha} \cdot \frac{(v')^{-1}\left(\frac{\varepsilon\alpha}{(\rho+\gamma+\varepsilon)(\rho+\gamma+\alpha)}\right)}{(v')^{-1}(1)} \right), \right. \\ \left. \frac{\varepsilon\alpha}{\varepsilon+\alpha} \cdot \left[\frac{\alpha}{\varepsilon}\gamma\,\overline{q} + \frac{\chi+\gamma}{\varepsilon+\alpha+\chi+\gamma} \right]^{-1} \right\}, \tag{16}$$

where $(\underline{q}, \overline{q})$ are the (lowest, highest) possible bond prices in any equilibrium, solved in closed form in Equations (17,18) below.

Proposition 3. *Given* (16)*, a stationary monetary equilibrium exists and is unique.*

Proof. See Appendix A.1. □

3.3 Characterization of equilibria

As has already been hinted by the asset market clearing equations (12), stationary equilibria can be classified into three regions. The region boundaries are illustrated in Figure 6 as functions of the money growth rate (γ), the asset market trading frequency (χ), and the relative supply of bonds to money (B/M). Formally, the classification is as follows:

Proposition 4. *Define the lowest and highest possible bond price, $\underline{q}$ and $\overline{q}$ respectively, as follows:*

$$\underline{q} = \frac{1}{2(\rho+\gamma)(\rho+\gamma+\varepsilon+\chi)(\rho+\varepsilon+\alpha)}\Big[\alpha\varepsilon + (\rho+\gamma+\varepsilon)^2 + (2\rho+2\gamma+\alpha+\varepsilon)\chi \ldots \tag{17}$$

$$+ \sqrt{-4\chi(\rho+\gamma)(\rho+\gamma+\varepsilon+\alpha)(\rho+\gamma+\varepsilon+\chi) + [(\rho+\varepsilon+\gamma)^2 + \chi(2\rho+2\gamma+\varepsilon) + \alpha(\varepsilon+\chi)]^2}\Big]$$

$$\overline{q} = \frac{1}{\rho+\gamma}\cdot\left[1 + \frac{\rho+\gamma+\chi}{\rho+\gamma+\varepsilon+\alpha}\right] \tag{18}$$

Then, there exist numbers b_{SR} and b_{BR}, where $b_{SR} > b_{BR}$, such that:

(a) Equilibrium satisfies $q = \underline{q}$ and $\psi_1 < 1$ (bond seellers are rationed) whenever $B/M > b_{SR}$;
(b) Equilibrium instead satisfies $q = \overline{q}$ and $\psi_0 < 1$ (bond buyers are rationed) whenever $B/M < b_{BR}$;
(c) Equilibrium satisfies $q = \tilde{q} = \chi M_0/(\chi B_1 + S)$ (nobody is rationed) otherwise.

Proof. See Appendix A.1. □

The economy behaves quite differently in each of these regions, as the analysis in the next section explains in detail.

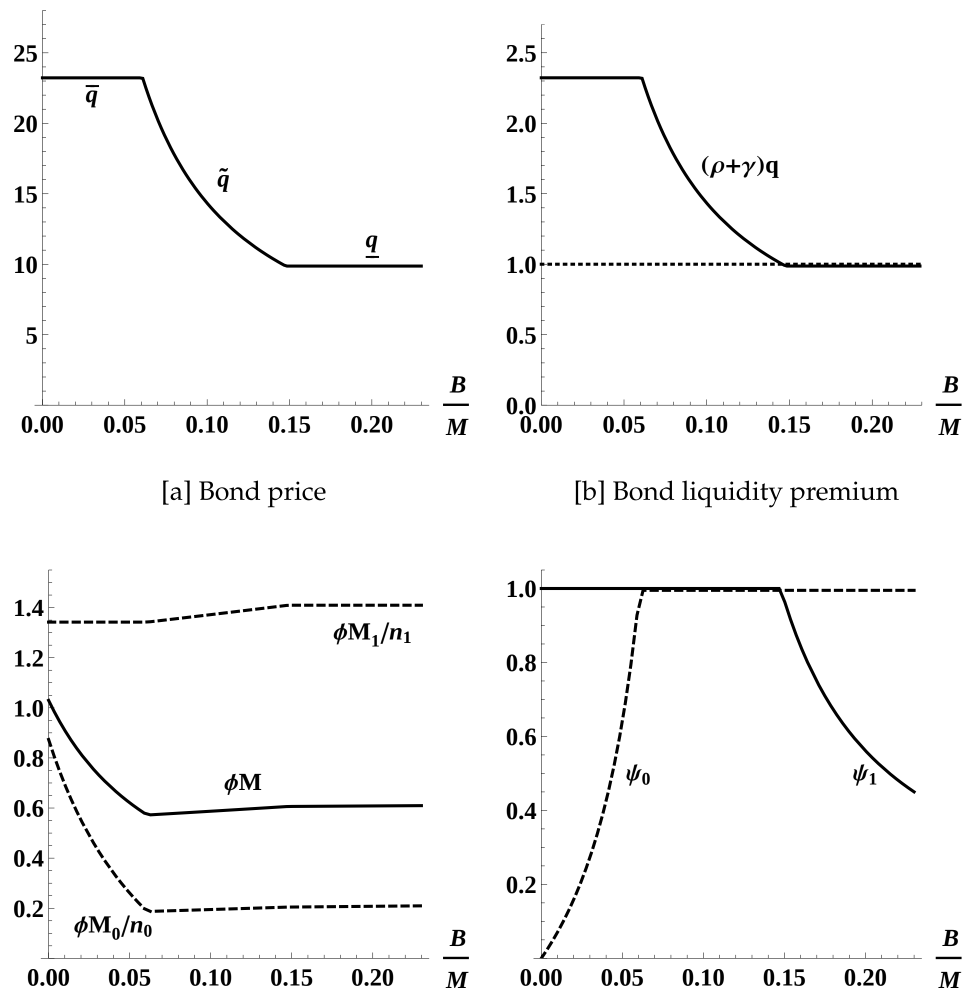
4 Analysis: monetary policy with frictional asset markets

Money is clearly not superneutral: inflation affects the value of holding money, which in turn affects labor supply and production. The comparative statics of the model with respect to inflation (γ) or asset market trading frequency (χ) are interesting but not the main focus of the paper, and are therefore relegated to Appendix A.5. We thus turn to the comparative statics with respect to the bond supply (B/M).

4.1 Long-run effects of the bond supply under monetary dominance

The comparative statics with respect to the bond supply are important because they help understand the twin roles these bonds play: they are better saving vehicles than money, and they provide indirect liquidity services because households can liquidate them when they expect to need money soon. To begin with, I assume that the money growth rate γ is fixed by monetary policy, and that the flow of lump-sum transfers, T, adjusts to satisfy the government budget constraint. Figure 7 shows the results.

As per Proposition 4, there are three regions of equilibrium. In the first, the supply of bonds by households in state 1 is too large for the demand by households in state 0; equilibrium is in this region if B/M is large, and in this case $q = \beta_1/\mu_1$ (the market price equals the reservation price of bond sellers) and $\psi_1 < 1$ (bond sellers are rationed). Within this region, changes in B/M have no effect on the economy.



[a] Bond price

[b] Bond liquidity premium

[c] Real balances per household: state-0, state-1, and average

[d] Trading probabilities (ψ_0 offset for visibility)

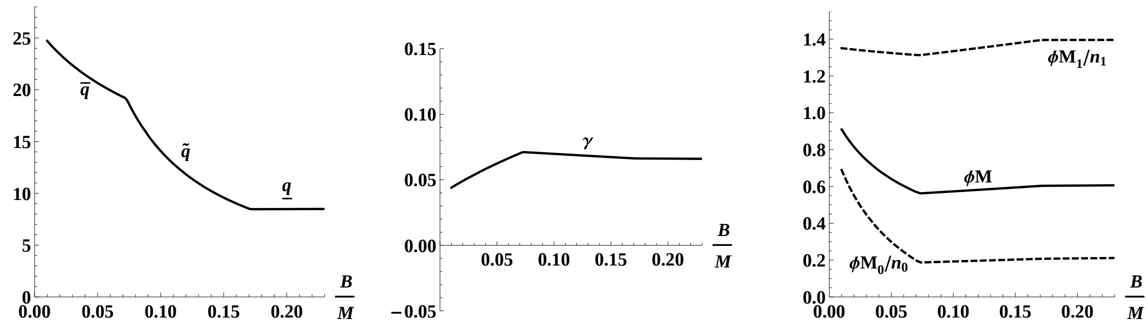
Figure 7: Comparative statics of the bond supply, relative to the money supply, under the assumption of monetary dominance. Total output is proportional to ϕM_1 (top line, panel [c]). Key parameters: $\rho = .05$ and $\gamma = \pi = .05$.

In the next region, the supply of bonds is intermediate, so that $q = \tilde{q}$. In this case, $\psi_0 = \psi_1 = 1$ (nobody is rationed) and an increase in bond supply directly decreases q. Using the Euler equations (6), we see that this decrease in q causes μ_0 to rise while μ_1 is unaffected; converting money into bonds becomes cheaper for households in state 0, and they are therefore willing to work harder and accumulate more money. By the goods market clearing equation (11), the extra production causes ϕM_1 to increase, and if the money supply is held fixed, this is achieved through a fall in the price level. The end result of an increase in bond supply in this region is lower prices, higher consumption and output, and higher welfare. The intuition is that these bonds provide a useful service: they help households in state 0 store their wealth in a way that avoids the inflation tax. As a result, such households accumulate wealth faster. Limiting the bond supply, conversely, drives down yields and may encourage households to invest in alternative assets with an offsetting positive effect on output (Herrenbrueck, 2014; Geromichalos and Herrenbrueck, 2017); but through the liquidity channel alone, a lower bond supply reduces output.

In the third region, the supply of bonds is so small that the demand by households in state 0 cannot be satisfied. Equilibrium is in this region if B/M is small, and in this case $q = \beta_0/\mu_0$ (the market price equals the reservation price of bond buyers) and $\psi_0 < 1$ (bond buyers are rationed). Because the bond price is finite (i.e., nominal yields are positive), it can still be affected by inflation and real shocks. But changes in B/M have no effect on the price level, production, or welfare, just like in the first region where the bond supply was large.

Hence, the region of low bond supply is a *liquidity trap*. Not because the yield on bonds was constrained by some kind of technical bound, whether zero in nominal terms or otherwise; instead, the bond price is constrained by households' willingness to pay for bonds with money. Or, according to the original definition (Keynes, 1936): "almost everyone prefers cash to holding a debt which yields so low a rate of interest". The fact that the possibility of a liquidity trap arises naturally in this model, with long-term bonds, positive bond yields, and flexible prices, illustrates what the liquidity trap does *not* require: short-term bonds, a technical zero lower bound, or sticky prices. (Nor that bonds be nominal; as shown in the working paper, the result and mechanism are much the same when bonds are real.) In comparison to the other regions, output and welfare are lowest in the liquidity trap.[11]

[11] Most results in this subsection – mainly, the generally positive liquidity premium, the downward-sloping demand for bonds, the weakly positive effect of bond supply on output, and the possible existence of a liquidity trap – match those derived in earlier models of imperfectly liquid bonds, including Berentsen and Waller (2011), Williamson (2012), Geromichalos and Herrenbrueck (2016a), Rocheteau, Wright, and Xiao (2018), Rocheteau, Weill, and Wong (2015), and others. They are here confirmed in a model with long-term bonds and persistent household heterogeneity, and they are important to review in any case since they help understand the new results in subsections 4.2-4.4. Other results are new – to wit, the negative liquidity premium when the bond supply is large, and the fact that positive long-term bond yields are consistent with a permanent liquidity trap.



[a] Bond price

[b] Implied inflation

[c] Real balances per household: state-0, state-1, and average

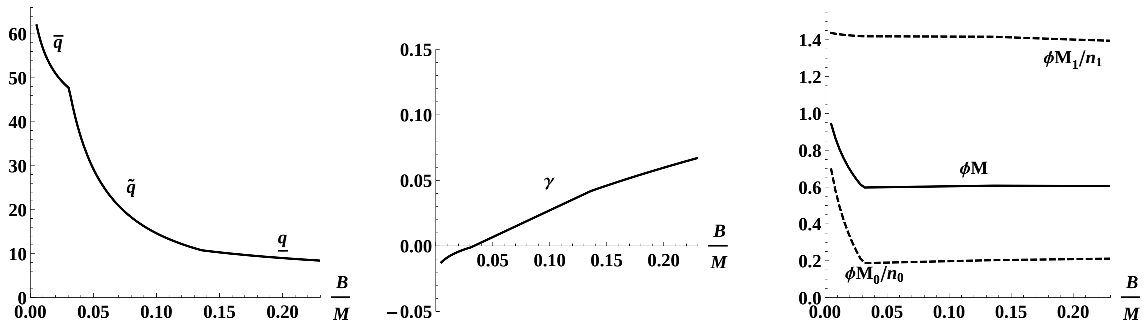
Figure 8: Comparative statics of the bond-money ratio, under fiscal dominance (inflation is endogenous) and a fixed real seigniorage revenue $\gamma\phi M = .04$. Total output is proportional to ϕM_1 (top line, panel [c]). Key parameters: $\rho = .05$, $\chi = 4$.

4.2 The liquidity trap, and fiscal dominance

In the liquidity trap region, a (tax-financed) change in the bond supply does not affect bond prices or output in the long run, as shown in Figure 7. However, higher expected inflation does have effects (see Appendix A.5): it increases the liquidity premium on bonds and, through a lower value of money, reduces production. Thus, an extended program of bond purchases, financed via a one-time expansion in the money supply, could still affect the economy even in the liquidity trap – through temporary inflation rather than through the lower stock of bonds – and in the opposite direction of what was, perhaps, intended.

However, all of the previous analyses assume monetary dominance: the government is committed to a certain growth rate of the money supply, and adjusts its long-run fiscal balance to satisfy the budget constraint. While common in monetary theory, this assumption is not always realistic. Government spending is often fixed in real terms rather than dollar units: e.g., ordering ten fighter jets for whatever the final price turns out to be, rather than ordering as many fighter jets as $100 million will buy. If the government is committed to a certain fiscal balance, possibly including debt service, and money growth adjusts as required, then we are in a regime of fiscal dominance. This case is especially realistic in an environment of low inflation and interest rates; an inflation-fighting central bank can raise rates to force the fiscal authority to spend less, but it cannot force the fiscal authority to spend more (Andolfatto, 2015). Then, the money growth rate is endogenously determined by the ratio of the fiscal deficit to the amount of money households are willing to hold.

Specifically, I look at two plausible assumptions. First, let total seigniorage revenue be fixed in real terms: $R \equiv \gamma\phi M$ being held constant. Then inflation is determined by:



[a] Bond price

[b] Implied inflation

[c] Real balances per household: state-0, state-1, and average

Figure 9: Comparative statics of the bond-money ratio, under fiscal dominance (inflation is endogenous), and a <u>fixed real structural deficit</u> $\phi T = -.02$. Total output is proportional to ϕM_1 (top line, panel [c]). Key parameters: $\rho = .05$, $\chi = 4$.

$$\gamma = \frac{R}{\phi M} = \frac{\alpha R}{n_0 h_0 + n_1 h_1} \frac{M_1}{M}$$

Alternatively, perhaps the real structural deficit (= transfer to households) is fixed in real terms ($D \equiv \phi T$ being held constant), but the costs of servicing the debt must be financed with seigniorage revenue.[12] Then inflation is determined by:

$$\gamma = \frac{\frac{D}{\phi} + B}{M + B}$$

With a representative household, there would not be much of a difference between fixing seigniorage in nominal or in real terms – as long as they match up, of course – when open-market purchases are conducted in a liquidity trap. But here, there is a big difference, because the *distribution* of money balances (M_0 vs M_1) affects the *level* of real balances households end up holding in equilibrium. For example, in the interior region, all households' real balances are positively related to the supply of bonds. But in the liquidity trap region, the level of real balances held by households in state 0 is strongly negatively related to the supply of bonds (see panel [c] of Figures 7-9).

The reason is that in this model, households in state 0 would prefer to convert money into bonds which have a better rater of return. When the supply of bonds is low, this happens more slowly ($\psi_0 < 1$). Therefore, the lower the aggregate stock of bonds, they higher is the proportion of the total stock of money held by households in state 0. The next step of the argument is crucial: it is not the total money stock that determines the price level

[12] A third possibility is considered by Andolfatto and Williamson (2015), who assume that the real lifetime value of outstanding government debt is kept fixed.

via goods market clearing, but the *money held by households looking to spend it* on goods (see Equation 11). This is the big lesson from household heterogeneity, which a representative-agent model would fail to capture. If the households about to spend money hold a lower share of it, then the velocity of circulation – and the aggregate price level – must be lower.

Consequently, a lower bond supply will increase the level of real balances households are willing to hold, and thereby reduce the inflation rate (panel [b] of Figures 8-9). Even if the government rebates the lower debt service cost to households as a transfer, so that the implied real seigniorage revenue remains constant (Figure 8), the increase in total real balances implicitly lowers the inflation rate. Naturally, if the lower debt service cost is not rebated to households but used to reduce the government's reliance on seigniorage (Figure 9), then inflation falls even more strongly, and not only in the liquidity trap.

4.3 Money is not neutral in the short run

Nominal assets are neutral in the long run (as long as the money-bond ratio is kept constant), but money injections are not neutral in the short run no matter what happens with bonds. A money transfer that gives everyone the same amount of cash – a "helicopter drop" – must raise prices, but thereby *compress* the distribution of real balances. The cash-rich will have a little bit less real cash, while the cash-poor will have a little bit more. Such a compression matters for real aggregates such as spending and production if the valuation of money and the propensity to spend it differ across households. The question is thus how the distributions of money holdings, money valuations, and propensities to spend money interrelate.

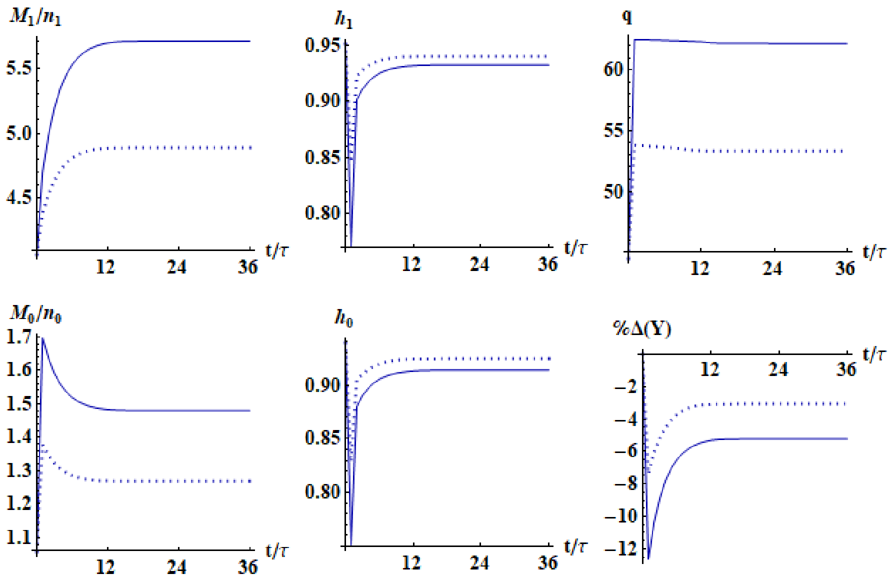
Here, the pattern is as follows. Households in state 1 have a higher propensity to spend money on consumption goods than those in state 0 (whose propensity to spend is zero). Knowing this, households in state 1 hold more money than those in state zero: they work harder, and they liquidate other assets for money. The distribution of money holdings *within* a state due to random trading histories is irrelevant, as everyone within a given state has the same marginal valuation of money and propensity to spend it.

Translated into technical terms: suppose that $N > 0$ new dollars of money are introduced lump-sum, i.e., $n_0 N$ dollars going to households in state 0 and $n_1 N$ dollars going to households in state 0. Denote the old price of money by ϕ and the new price of money by $\hat{\phi}$, and by $(\hat{M}_0 = M_0 + n_0 N, \hat{M}_1 = M_1 + n_1 N)$ the post-transfer money totals.

Proposition 5. *In a stationary equilibrium satisfying condition* (16)*, the following are true:*

$$\frac{M_0}{n_0} < \frac{M_1}{n_1}, \qquad \textit{and:} \qquad \frac{\hat{M}_1}{\hat{M}_0} = \frac{n_1}{n_0} \cdot \frac{M_1/n_1 + N}{M_0/n_0 + N} \;<\; \frac{n_1}{n_0} \cdot \frac{M_1/n_1}{M_0/n_0} = \frac{M_1}{M_0}.$$

Proof. The latter inequality follows from the former, which in turn follows from $M_1/M > n_1$. This was proved to follow from condition (16) as part of the proof of Proposition 3. □



Continuous line: 40% increase in the money stock. Dotted line: 20% increase

Figure 10: Dynamics after a one-time helicopter drop. The drop is unanticipated in period 0 and happens in period 1. Showing: the money held on average per household in state 0 and 1, the labor supply by households in state 0 and 1 (first-best: 1), and the price of bonds (fundamental: $1/\rho = 33.3$), all in levels. Finally, output is shown in percent deviations from its initial level. The steady-state inflation rate is held at zero.

As a result, a helicopter drop of money will shift purchasing power away from state-1 households and towards state-0 households. Naturally, the increase in the money supply raises the general price level ($1/\phi$), but by how much? If it rises in proportion to the money stock ($\phi/\hat{\phi} = 1 + N/M$), then ϕM_1 falls (since state-1 households held more than the average amount of money, and their share went down), and we have a lack of demand in the goods market. If it rises only enough to keep ϕM_1 constant, then ϕM_0 exceeds its steady-state level, so we must anticipate excess inflation along the transition path. This decreases the value of money and motivates households to work less and supply fewer goods, so we have a lack of supply in the goods market. (For an illustration, see Figure 10 which shows a discretized version of the dynamic model described in Appendix A.2.)

The only solution lies somewhere in the middle, depending on the elasticity of supply: the price level must undershoot its long-term path, and along the transition path the goods market clearing equation is satisfied at a lower level of consumption and production. Correspondingly, state-0 households have excess purchasing power compared to the steady state. What do they spend it on? Bonds, in the bond market. Thus – as long as equilibrium is not in the liquidity trap region where the bond price is already maximal – the helicopter drop also increases the bond price q along the transition path. Equivalently, it lowers real bond yields.[13]

Of course, in the real world this is not the end of the story: this fall in yields may stimulate investment and output, *indirectly*, as other assets such as capital or equity are now more valuable to create (which is the focus of the working paper version of this paper). But in contrast to traditional wisdom, the helicopter drop also has a *negative direct effect* on consumption demand, output, and welfare, as money is less efficiently distributed.

As Figure 10 also shows, an increase in the money supply is not even neutral in steady state because bonds are nominal, and their supply is not being expanded to match. This increase in the money-bonds ratio lowers the *real* supply of bonds. The implication is a higher long-run bond price and lower long-run output (as shown in detail in Section 4.1 below). This would be different if the bonds were real, as the long-run ratio of real balances to real bonds would be unaffected by the one-time helicopter drop.

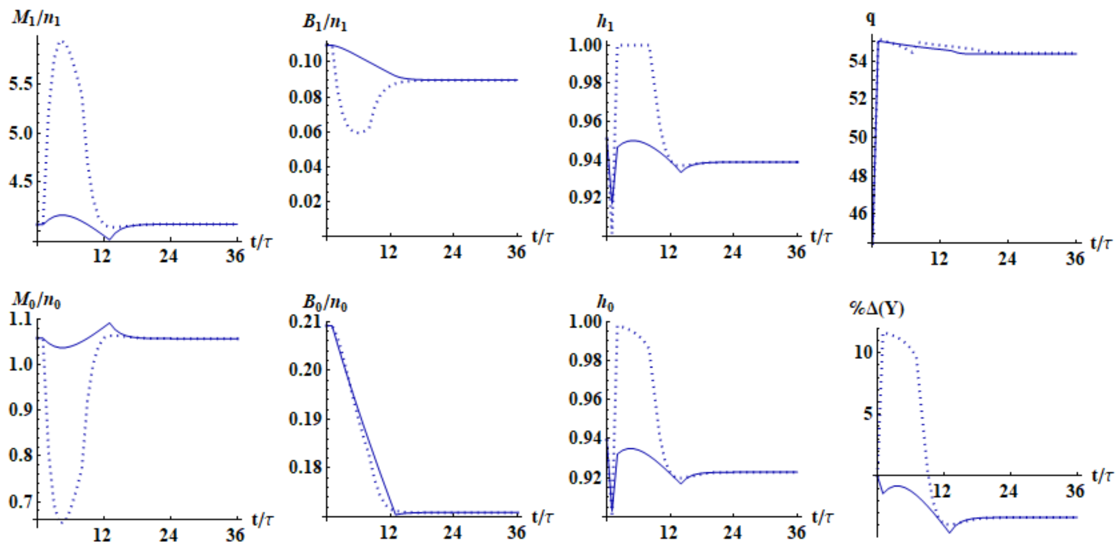
All of the distributional results follow from the fact that in this model, the quantity of money someone holds and their average propensity to spend this money are positively correlated. This is due to two assumptions: the heterogeneous arrival of spending opportunities (α versus 0, due to the state cycle), together with the linear value function that makes the within-state distribution of money irrelevant. Most models of heterogeneous money holdings have made the opposite assumptions: all households have the same arrival rate of spending opportunities, but households have concave utility of consumption, which induces a downward-sloping marginal utility of money (Berentsen, Camera, and Waller, 2005; Molico, 2006; Chiu and Molico, 2010; Rocheteau, Weill, and Wong, 2015). This gives rise to a risk-sharing channel: the cash-rich households value a marginal dollar less than the cash-poor ones, thus compressing the distribution of real balances – via a helicopter drop – generally results in higher spending, production, and welfare. (But not always: e.g, see Jin and Zhu (2017) where the motivation to work falls by enough to offset the increased ability to spend, and Kam and Lee (2017) where the estimated inflation tax dominates the gains from redistribution). In reality, we should expect both channels to coexist, but an empirical analysis of which channel dominates is beyond the scope of the present paper.

The fundamental question is: why do we think some people hold more money than others? By accident, because they tried to spend it but failed? Or by intention, because they anticipate spending it soon?

4.4 Helicopter drops versus open-market purchases

Finally, I analyze a temporary intervention in the asset market where $S < 0$ for a finite period of time (an open-market purchase of illiquid bonds). For simplicity, assume that the economy

[13] Alvarez and Lippi (2014) derive a very similar result – after a money injection, the price level undershoots and interest rates fall – but in an endowment economy where there is no effect on output.

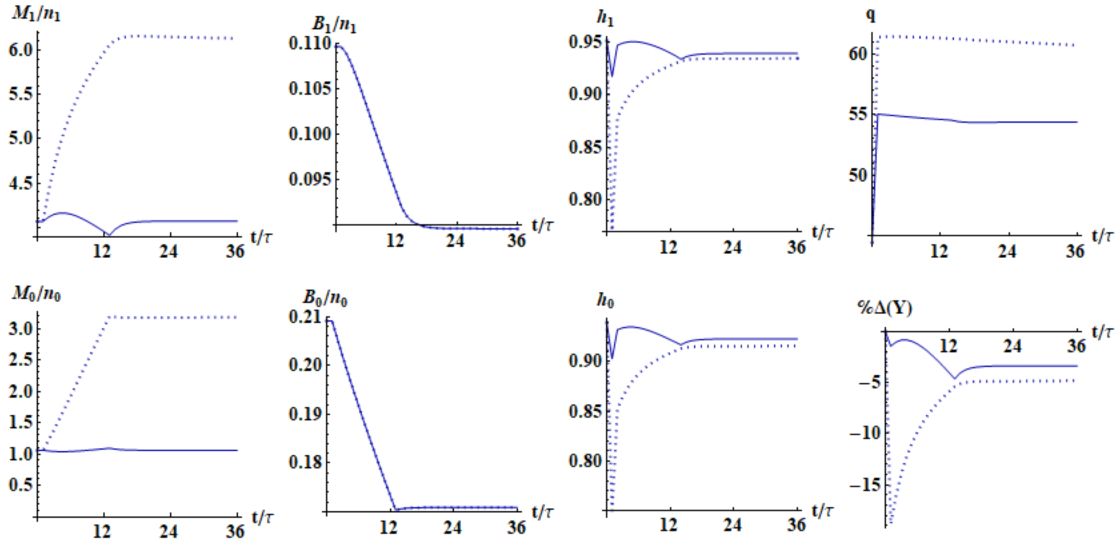


Continuous line: 20% decrease in the bond stock, purchased uniformly over 12 periods
Dotted line: "tapered" purchase, starting at twice the pace and declining to 0 linearly

Figure 11: Dynamics of a 12-period bond purchase program, financed with lump-sum taxes. The program is unanticipated in period 0 and fully understood from period 1 on. Showing: the money and bonds held on average per household in state 0 and 1, the labor supply by households in state 0 and 1 (first-best: 1), and the price of bonds (fundamental: $1/\rho = 33.3$), all in levels. Finally, output is shown in percent deviations from the initial level. Steady-state inflation is held at zero.

was previously in a stationary equilibrium with zero inflation and no expectations of intervention, and that the asset market was in the interior region where $q = \tilde{q} = M_0/B_1$. Then, the government intervenes in the frictional bond market to buy bonds for money ($S < 0$), driving up bond prices to $q = \chi M_0/(\chi B_1 + S)$. During the intervention, M_1 is higher relative to the old steady state, M_0 and B_1 are lower, and B_0 is unaffected (to a first approximation). The rise in M_1 implies more consumption and higher output during the intervention, but also a rise in the price level to absorb some of the extra demand.

Once the intervention has concluded, however, and $q = M_0/B_1$ holds again, the temporary fall in ϕM_0 will *depress* bond prices along the transition path to the new steady state. Because the supply of bonds available to households is lower in the new steady state than in the initial one, bond prices q will ultimately be higher. Thus there are two opposing forces on q after the intervention: the need to converge to a higher level in the new steady state, and the lack of demand for bonds due to a temporary depression of ϕM_0. Either one could prevail quantitatively. In summary, an open-market purchase of illiquid assets causes lower yields both during the intervention and in the long run, but possibly not along the entire transition path. In addition, the purchase has a positive direct effect on output in the short



Continuous line: bond purchases are financed with lump-sum taxes
Dotted line: bond purchases are financed with newly printed money

Figure 12: Dynamics of a 12-period bond purchase program, a 20% decrease in the bond stock purchased uniformly over 12 periods. The program is unanticipated in period 0 and fully understood from period 1 on. Showing: the money and bonds held on average per household in state 0 and 1, the labor supply by households in state 0 and 1 (first-best: 1), and the price of bonds (fundamental: $1/\rho = 33.3$), all in levels. Finally, output is shown in percent deviations from the initial level. Steady-state inflation is held at zero.

run because real balances are more concentrated in the hands of those planning to spend them, but a negative direct effect on output in the long run because bonds perform a useful service in this economy.

The dynamic equilibria following three variants of a bond purchase program are illustrated in Figures 11-12.[14] It is instructive to contrast the effects of these programs with those of a helicopter drop analyzed above. Recall that a helicopter drop compresses the distribution of real balances, shifting purchasing power into the hands of people planning to buy assets (state-0 households) and out of the hands of people planning to buy goods (state-1 households). An open-market purchase does the opposite. It shifts purchasing power away

[14] Specifically, the experiment is as follows. In period 0, the economy is in steady state. In period 1, a bond purchase program is announced, whereby the government will buy back 20% of the outstanding bond supply in the asset market. While the announcement is a complete surprise, the resulting equilibrium dynamics are immediately understood. I consider three variations of the purchases: first, a "uniform, tax-financed" program, where the flow of purchases is constant over 12 periods (i.e., purchasing $20\%/12 \approx 1.66\%$ of outstanding bonds every period), and lump-sum taxes keep the money stock constant throughout; second, a "tapered, tax-financed" program, where the flow of purchases starts fast (i.e., purchasing $2 \times 20\%/12 \approx 3.33\%$ of bonds in period 1) and declines to zero linearly (i.e. purchasing 0.3% of bonds in period 11 and nothing thereafter); third, a "uniform, inflation-financed" program where the flow of purchases is uniform as in the first variant, but lump-sum transfers remain constant and all purchases are financed with newly printed money.

from asset demand and towards goods demand, although total asset demand may still be increased when we add the open-market purchase itself ($S < 0$) to the remaining private demand. The direct effects on output therefore go in opposite directions, but both interventions cause lower bond yields.

In the short run, the bond purchase program always increases the prices of bonds and directs purchasing power towards state-1 households and away from state-0 households. Labor supply and output fall on impact (because the lower new steady state is anticipated), but then experience a temporary boom due to the extra demand for consumption goods. The boom is especially strong under a tapered, tax-financed program, strong enough to temporarily offset the contractionary long-term effect of a lower bond supply.

5 Conclusion

The model contributes to the theory of money and asset markets in important ways. For one, it is parsimonious: the only frictions are trading delays in asset markets and the fact that money is occasionally necessary to purchase consumption goods. Households are heterogeneous and their individual portfolios depend on history, but this is a feature rather than a friction. Financial assets are safe and long-term, thereby abstracting from risk premium or yield curve effects.[15] Even in this simple framework, money is not neutral in the short run, government intervention in asset markets has persistent effects, and the supply of illiquid assets matters for the macroeconomy.

By taking asset markets seriously, the model offers new insights into the effects of monetary policy. In contrast to the overwhelming majority of the literature, open-market operations are modeled realistically as intervention in asset markets (rather than as directly manipulating households' budget constraints), and the difference matters. First, the intervention reallocates purchasing power among asset buyers and asset sellers; just as an open-market purchase stimulates private demand for goods, it crowds out private demand for illiquid assets. Second, when assets are already scarce enough to push the economy into a liquidity trap, further purchases slow down the circulation of money. This fact can cause disinflation and higher real interest rates, defeating the intent of the intervention.

In this paper, I restrict attention to a two-asset economy: money and a consol bond. One may also be interested in a capital asset which is *both* tradable in frictional asset markets and useful in production. Including such a capital asset in the model unlocks lessons about the effect of monetary policy on investment dynamics. As I show in the working paper ver-

[15] Finite-term bonds can be liquidated in two ways: by selling them in the market, or by letting them mature. Since short-term bonds mature sooner than long-term bonds, they have inherently different liquidity properties. Geromichalos, Herrenbrueck, and Salyer (2016) analyze the implications of this fact for bond prices and the yield curve, and Williamson (2016) studies open market operations designed to twist the yield curve.

sion of this paper (Herrenbrueck, 2014), when bonds and capital are imperfect substitutes as assets then lower bond yields will stimulate investment and, thereby, output. Whether the stimulative indirect effect is strong enough to offset the contractionary direct effect depends mainly on two elasticities: the elasticity of substitution between assets as stores of value, and the elasticity of labor supply (which determines the strength of the inflation tax). This alternative is interesting since almost all 'textbook' discussions of the effect of monetary policy on the economy are limited to this *investment channel* – lower interest rates promote investment, through some form of substitution between bonds and capital – but neglect the *liquidity channel* studied here.

This model is mainly a contribution to the positive theory of monetary policy implementation. Because of the simple structure of the model, the only possible welfare loss comes from $h_0, h_1 < h^{FI}$; households' labor effort, and thus production, being below the first-best level. Thus, output is a sufficient statistic for welfare, higher money stocks and faster money growth both tend to be bad policies, and the Friedman rule achieves the first-best. However, the list of realistic model ingredients that could alter this result is extensive: if lump-sum taxes are not available (Hu, Kennan, and Wallace, 2009; Andolfatto, 2013), if the government has an advantage in providing public goods and needs taxes to finance them, if a fraction of money is spent on socially worthless activity (as assumed by Williamson, 2012), or if some agents have market power (Herrenbrueck, 2017), then the normative implications of the model could be different without affecting the model's ability to explain how interventions in asset markets affect asset prices and the broader economy.

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Appendix

A.1 Proofs of statements

Proof of Proposition 1. First, write the Bellman equations (2) as a vector functional equation $T[(f_0, f_1)] = (f_0, f_1)$, where:

$$T\left[\begin{array}{c} f_0 \\ f_1 \end{array}\right](z^M, z^B) = \left(\begin{array}{l} \sup \int_0^\infty e^{-(\rho+\varepsilon+\chi)t}\Big[-v(h_t) + \varepsilon\, f_1(z_t^M, z_t^B) + \chi\, f_0(z_t^M + q_t s_{0,t}, z_t^B - s_{0,t})\Big]dt \\ \sup \int_0^\infty e^{-(\rho+\alpha+\chi)t}\Big[-v(h_t) + \alpha\left[c_t + f_0(z_t^M - c_t, z_t^B)\right] + \chi\, f_1(z_t^M + q_t s_{1,t}, z_t^B - s_{1,t})\Big]dt \end{array}\right)$$

with respect to left-continuous plans for flow labor, consumption, real balances, and bonds, subject to (3). This is almost exactly the problem from Theorem 1 of Rocheteau, Weill, and Wong (2015) (RWW, henceforth), with only three differences: (1) it is a two-dimensional vector equation; (2) the problem is only weakly concave since the utility of consumption is linear; (3) there is a second state variable (bonds) in addition to real balances. However, every step of their proof applies, except those relating to the second derivative of the value function (which are the only ones for which strict concavity is used). There is no need to prove that the bond holdings will not be changed too quickly (as RWW prove for money holdings), because at a generic point in time the household can do nothing to adjust its bond holdings. Adjusting bond holdings is only possible in the instant when the household has access to the asset market, but these instants are handled by the transition term $\chi W_i(\dots)$ in the household's problem (since the integral is over the time spent *until* the arrival of either the transition shock or the asset market access shock).

The Hamilton-Jacobi-Bellman equations are exactly analogous to those from RWW, adding only the state transition terms, asset trading shock terms, and the marginal value of bonds times the change in the bond stock $\partial_B W_i(z^M, z^B) \cdot [-\gamma z^B]$. $\square$

Proof of Proposition 2. Begin with the HJB equations (4) from Proposition 1. According to that proposition, they have a unique solution. Thus, to prove linearity, it suffices to guess a linear solution and verify that it indeed solves the HJB equations. I impose Equation (5) and add Lagrange multipliers for the spending constraints ($\underline{\lambda}_i$ for $s_i \geq -z^M q$, $\overline{\lambda}_i$ for $s_i \leq z^B$, and η for $c \leq z^M$). Then, the HJB equations become:

$$(\rho+\varepsilon+\chi)\left[W_0(0,0) + \mu_0 z^M + \beta_0 z^B\right] = \dots$$
$$\max_{\substack{h_0, s_0, \\ \underline{\lambda}_0, \overline{\lambda}_0}} \Bigg\{ -v(h_0) + \varepsilon\left[W_1(0,0) + \mu_1 z^M + \beta_1 z^B\right] + \chi\left[W_0(0,0) + \mu_0(z^M + q s_0) + \beta_0(z^B - s_0)\right]$$
$$+ \mu_0\left(z^B + h_0 - \gamma z^M + \Upsilon\right) - \beta_0 \gamma z^B + \underline{\lambda}_0\left(s_0 + \frac{z^M}{q}\right) + \overline{\lambda}_0\left(z^B - s_0\right)\Bigg\}$$

$$
\begin{aligned}
&(\rho+\alpha+\chi)\left[W_1(0,0)+\mu_1 z^M+\beta_1 z^B\right]=\dots \\
&\qquad \max_{\substack{h_1,s_1,c\\ \underline{\lambda}_1,\overline{\lambda}_1,\eta}}\Bigg\{-v(h_1)+\alpha\left[c+W_0(0,0)+\mu_0(z^M-c)+\beta_0 z^B\right]+\chi\left[W_1(0,0)+\mu_1(z^M+qs_1)+\beta_1(z^B-s_1)\right] \\
&\qquad\qquad +\mu_1\left(z^B+h_1-\gamma z^M+\Upsilon\right)-\beta_1\gamma z^B+\underline{\lambda}_1\left(s_1+\frac{z^M}{q}\right)+\overline{\lambda}_1\left(z^B-s_1\right)+\eta\left(z^M-c\right)\Bigg\}
\end{aligned}
$$

Take first-order conditions to obtain:

$$
h_i: \qquad 0=-v'(h_i)+\mu_i \tag{A.2}
$$

$$
s_i: \qquad 0=\chi\left(\mu_0 q-\beta_0\right)+\underline{\lambda}_i-\overline{\lambda}_i \tag{A.3}
$$

$$
c: \qquad 0=\alpha(1-\mu_0)-\eta \tag{A.4}
$$

Equation (A.2) immediately delivers the solution for labor supplies (part (b)), and the solution for asset trading demands follows from Equation (A.3) together with the complementary slackness conditions: $\underline{\lambda}_i>0$ implies $s_i=-z^M/q$, and $\overline{\lambda}_i>0$ implies $s_i=z^B$.

Differentiating the HJB equations with respect to z^M and z^B and applying the envelope theorem yields:

$$
\begin{aligned}
(\rho+\varepsilon+\chi)\mu_0 &= \varepsilon\mu_1+\chi\mu_0-\gamma\mu_0+\frac{\underline{\lambda}_0}{q} \\
(\rho+\varepsilon+\chi)\beta_0 &= \varepsilon\beta_1+\chi\beta_0-\gamma\beta_0+\mu_0+\overline{\lambda}_0 \\
(\rho+\alpha+\chi)\mu_1 &= \alpha\mu_0+\chi\mu_1-\gamma\mu_1+\frac{\underline{\lambda}_1}{q}+\eta \\
(\rho+\alpha+\chi)\beta_1 &= \alpha\beta_0+\chi\beta_1-\gamma\beta_1+\mu_1+\overline{\lambda}_1
\end{aligned}
$$

From (A.4), we can immediately substitute $\eta=\alpha(1-\mu_0)$. For the asset trade Lagrange multipliers, suppose that $\underline{\lambda}_i>0$. Then, by complementary slackness, $s_i=-z^M/q$ and therefore $\overline{\lambda}_i=0$, which implies $\underline{\lambda}_i=\chi\left(\beta_i-\mu_i q\right)$. The only alternative is $\underline{\lambda}_i=0$; thus, we can substitute $\max\{\beta_i-\mu_i q,0\}$ in place of $\underline{\lambda}_i$. Analogously, we can substitute $\max\{\mu_i q-\beta_i,0\}$ in place of $\overline{\lambda}_i$. After rearranging, we obtain the Euler equations (6), as claimed.

These equations are independent of the quantities (z^B,z^M); since the quantities are arbitrary, the functions $(W_i(0,0)+\mu_i z^M+\beta_i z^B)_{i=0,1}$ where $(\mu_i,\beta_i)_{i=0,1}$ satisfy (6) solve the HJB equations (4). By uniqueness, this is the only solution, and linearity is verified.

To confirm that all real balances are spent on consumption when the opportunity arises, we need to verify $\mu_0<1$. Since this is part of the inequalities in part (d), I digress to consider them all together. Take the set of Euler equations (6), and begin with assuming that $\chi=0$. In that case, $\mu_0<\mu_1<1$ is obvious given that $\gamma>-\rho$. Next, add the equations for β_0 and β_1 and arrange them to yield:

$$(\rho+\gamma+\varepsilon+\alpha)(\beta_1-\beta_0)=\mu_1-\mu_0,$$

establishing $\beta_0<\beta_1$. Finally, the equations for β_0/μ_0 and β_1/μ_1 can be solved explicitly:

$$\frac{\beta_0}{\mu_0}=\frac{1}{\rho+\gamma}+\frac{1}{\rho+\gamma+\varepsilon+\alpha}\qquad\text{and}\qquad\frac{\beta_1}{\mu_1}=\frac{1}{\rho+\gamma}+\frac{1}{\rho+\gamma+\varepsilon+\alpha}-\frac{1}{\rho+\gamma+\varepsilon}$$

As $\gamma>-\rho$, the fact that $\beta_0/\mu_0>1/(\rho+\gamma)>\beta_1/\mu_1$ follows, and the implication is that households in state 0 (1) want to buy (sell) bonds, at least when $\chi=0$.

Now, suppose that there exists a $\chi>0$ such that $\beta_0/\mu_0<\beta_1/\mu_1$; in that case, by continuity there must also exist a $\bar{\chi}>0$ such that $\beta_0/\mu_0=\beta_1/\mu_1$ exactly. Then, the trade surplus terms must be equal for agents in all states no matter the bond price q. Adding equal terms to both the 0- and the 1-equations does not change their ranking relative to when all trade surpluses are zero, and we have already established that in this case the inequality $\beta_0/\mu_0>\beta_1/\mu_1$ holds. So the existence of $\bar{\chi}$ is contradicted and the inequality must hold for any $\chi\in[0,\infty)$.

Now turn to the remaining two inequalities. Divide the equation for β_0 through by μ_0:

$$(\rho+\gamma)\frac{\beta_0}{\mu_0}=1+\varepsilon\frac{\beta_1-\beta_0}{\mu_0}+\chi\max\left\{q-\frac{\beta_0}{\mu_0},\,0\right\}$$

As the $\max$-term is non-negative, and as $\beta_0<\beta_1$ (proven above), the claim $\beta_0/\mu_0>1/(\rho+\gamma)$ follows.

Finally, assume that $q=\beta_1/\mu_1$, consider the equation for β_1, and rearrange it:

$$(\rho+\gamma)\beta_1=\mu_1+\alpha(\beta_0-\beta_0)\qquad\Rightarrow\qquad(\rho+\gamma)\frac{\beta_1}{\mu_1}=1-\alpha\frac{\beta_1-\beta_0}{\mu_1}$$

As $\beta_0<\beta_1$, the claim $\beta_1/\mu_1<1/(\rho+\gamma)$ follows: when bonds are priced at the lowest level possible (i.e. when they are abundant), then their price reflects an illiquidity discount.

Finally, return to part (c) and consumption demand. By inequality $\mu_0<1$, the Lagrange multiplier $\eta=\alpha(1-\mu_0)$ is positive and all real balances must be spent on consumption: $c=z^M$. □

Proof of Proposition 3. The strategy of the proof is simple: set aside the asset market clearing condition, let q be exogenous, and construct the rest of the equilibrium. Then, close the loop by solving for asset market clearing. What makes this work is the fact that all equations other than asset market clearing are linear in equilibrium variables other than ψ_0,ψ_1, and q, or determined in a block, such as h_0 and h_1. Furthermore, the costate equations (6) on the one hand, and the asset flow equations (14) together with goods market clearing and the government budget on the other hand, form distinct blocks that we can solve separately.

First, costates. By the second inequality from (9), a bond price q can only be part of an equilibrium if $\beta_1/\mu_1 \leq q \leq \beta_0/\mu_0$. Taking q within these bounds as given, the equations form a linear block:

$$
\begin{aligned}
(\rho+\gamma+\varepsilon+\chi)\,\mu_0 &= \varepsilon\mu_1 + \chi\,\beta_0/q \\
(\rho+\gamma+\alpha)\,\mu_1 &= \alpha \\
(\rho+\gamma+\varepsilon)\,\beta_0 &= \mu_0 + \varepsilon\beta_1 \\
(\rho+\gamma+\alpha+\chi)\,\beta_1 &= \mu_1 + \alpha\beta_0 + \chi\,\mu_1\,q
\end{aligned}
$$

with a unique solution for given q. Now, since $\beta_1/\mu_1 \leq q \leq \beta_0/\mu_0$, there are three possible outcomes: $q = \underline{q} = \beta_1/\mu_1$ (the lower bound), the interior, and $q = \overline{q} = \beta_0/\mu_0$ (the upper bound). Setting $q = \beta_1/\mu_1$, we can solve a quadratic equation and select the larger of two results (the only one guaranteed to be positive):

$$
\begin{aligned}
\underline{q} &= \frac{\beta_1}{\mu_1} \\
&= \frac{1}{2(\rho+\gamma)(\rho+\gamma+\varepsilon+\chi)(\rho+\varepsilon+\alpha)}\Big[\alpha\varepsilon + (\rho+\gamma+\varepsilon)^2 + (2\rho+2\gamma+\alpha+\varepsilon)\chi \ldots \\
&\quad + \sqrt{-4\chi(\rho+\gamma)(\rho+\gamma+\varepsilon+\alpha)(\rho+\gamma+\varepsilon+\chi) + [(\rho+\varepsilon+\gamma)^2 + \chi(2\rho+2\gamma+\varepsilon) + \alpha(\varepsilon+\chi)]^2}\Big]
\end{aligned}
$$

which is Equation (17). And setting $q = \beta_0/\mu_0$, we can solve for the upper bound (Equation 18), which is simply:

$$
\overline{q} = \frac{\beta_0}{\mu_0} = \frac{1}{\rho+\gamma} \cdot \left[1 + \frac{\rho+\gamma+\chi}{\rho+\gamma+\varepsilon+\alpha}\right]
$$

Contrary to the model with real bonds (solved in the working paper), in the model with nominal bonds the upper bound $\overline{q}$ is always finite.

In steady state, the bond supply must expand at the rate of money growth: $S = \gamma B$. The candidate bond price therefore equals: $\tilde{q} = \chi M_0/(\chi B_1 + \gamma B)$. Thus, we need to solve for the rest of the equilibrium in three cases: specifically, $q = \underline{q}$, $q = \tilde{q} \in (\underline{q}, \overline{q})$, and $q = \overline{q}$. Start in the interior:

<u>Interior case:</u> Suppose that $\tilde{q} \in (\underline{q}, \overline{q})$ (to be verified). Then, $\psi_0 = \psi_1 = 1$ and we can solve directly for steady-state bond holdings:

$$
B_0 = \frac{\alpha+\chi+\gamma}{\varepsilon+\alpha+\chi+\gamma}\,B \qquad \text{and} \qquad B_1 = \frac{\varepsilon}{\varepsilon+\alpha+\chi+\gamma}\,B \tag{A.6}
$$

Next, take the flow equation for M_0 in steady state and use the government's budget constraint to substitute T, to obtain:

$$\gamma M_0 = B_0 + n_0 \left(\frac{h_0}{\phi} + \gamma M + (\gamma q - 1) B \right) - (\varepsilon + \chi) M_0$$

And using goods market clearing to substitute for ϕ:

$$(\varepsilon + \chi + \gamma) M_0 = B_0 + n_0 \gamma M + n_0 (\gamma q - 1) B + \frac{n_0 h_0}{n_0 h_0 + n_1 h_1} \alpha M_1$$

Substitute $M_1 = M - M_0$, substitute B_0 from above, and rearrange:

$$\frac{M_0}{M} = \frac{1}{\varepsilon + \frac{n_0 h_0}{n_0 h_0 + n_1 h_1} \alpha + \chi + \gamma} \left[n_0 \gamma + \frac{n_0 h_0}{n_0 h_0 + n_1 h_1} \alpha + \left(\frac{\alpha + \chi + \gamma}{\varepsilon + \alpha + \chi + \gamma} + n_0 (\gamma q - 1) \right) \frac{B}{M} \right]$$

From now on, simplify notation by defining $\tilde{n} \equiv n_0 h_0 / (n_0 h_0 + n_1 h_1)$. Since the labor supplies are endogenous, so is $\tilde{n}$. However, it is narrowly bounded: because $\mu_0 < \mu_1 = \alpha / (\rho + \gamma + \alpha)$, and $(v')^{-1}$ is an increasing function, we have $\tilde{n} \in [0, n_0]$.

Finally, consider the candidate bond price $\tilde{q} = \chi M_0 / (\chi B_1 + \gamma B)$:

$$\begin{aligned}\tilde{q} &= \frac{M_0}{M} \cdot \frac{M}{B} \cdot \frac{B}{B_1 + (\gamma/\chi) B} \\ &= \frac{1}{\varepsilon + \tilde{n}\alpha + \chi + \gamma} \left[(n_0 \gamma + \tilde{n} \alpha) \frac{M}{B} + \frac{\alpha + \chi + \gamma}{\varepsilon + \alpha + \chi + \gamma} + n_0 (\gamma q - 1) \right] \Big/ \left(\frac{\varepsilon}{\varepsilon + \alpha + \chi + \gamma} + \frac{\chi}{\gamma} \right)\end{aligned}$$

Set $\tilde{q} = q$, and solve:

$$\tilde{q} = \left[\frac{\varepsilon (\varepsilon + \tilde{n} \alpha + \chi + \gamma)}{\varepsilon + \alpha + \chi + \gamma} + \gamma \left(n_1 + \frac{\varepsilon + \tilde{n} \alpha + \gamma}{\chi} \right) \right]^{-1} \cdot \left[\frac{\alpha + \chi + \gamma}{\varepsilon + \alpha + \chi + \gamma} - n_0 + (n_0 \gamma + \tilde{n} \alpha) \frac{M}{B} \right] \quad \text{(A.7)}$$

after some algebraic simplification. This term only depends on exogenous variables, plus the labor supplies h_0 and h_1 (via $\tilde{n}$). But $h_1 = (v')^{-1}[\alpha / (\rho + \gamma + \alpha)]$ in steady state, which is also exogenous, and $h_0 = (v')^{-1}[\mu_0]$. Now, μ_0 is clearly decreasing in q (making bonds more expensive to buy discourages state-0 households from holding money that they could spend on bonds), and v' and its inverse are increasing functions, so $\tilde{n}$ must be decreasing in q. The presence of the $\tilde{n}$-term in the denominator makes it hard to prove uniqueness in the abstract, but since $\tilde{n} \in (0, n_0)$ (and $n_0 = \alpha / (\varepsilon + \alpha)$), the solution for $\tilde{q}$ must be unique as long as α is small enough relative to the other terms and χ is large enough.

However, this establishes the equilibrium only if $\tilde{q}$ lies inside the interval $[\underline{q}, \overline{q}]$. We will consider the alternatives next.

Upper bound case: Suppose that $\tilde{q} > \overline{q}$. Then $\psi_1 = 1$ but $\psi_0 < 1$; in fact, we have:

$$\chi \psi_0 M_0 = \overline{q} (\chi B_1 + \gamma B) = \left(\frac{\varepsilon \chi}{\varepsilon + \alpha + \chi + \gamma} + \gamma \right) \overline{q} \, B$$

In this case, $\psi_1 = 1$ implies that the bond holdings in steady state still satisfy Equation (A.6).

M_1 is still determined by goods market clearing, and h_0 and h_1 (and, thereby, $\tilde{n}$) are determined by the solution to the costates block at $q = \overline{q}$. The flow equation for M_0 in steady state now becomes:

$$\gamma M_0 = B_0 + \tilde{n}\alpha(M - M_0) + n_0\left(\gamma M + (\gamma q - 1)B\right) - \varepsilon M_0 - \overline{q}(\chi B_1 + \gamma B)$$

Substitute B_0 and B_1:

$$(\varepsilon + \tilde{n}\alpha + \gamma)M_0 = (n_0\gamma + \tilde{n}\alpha)M + \left[\frac{\alpha + \chi + \gamma}{\varepsilon + \alpha + \chi + \gamma} - n_0 - \left(\frac{\varepsilon\chi}{\varepsilon + \alpha + \chi + \gamma} + n_1\gamma\right)\overline{q}\right]B,$$

implying a unique solution for M_0 in terms of exogenous or already-solved-for variables. (If the implied result is $M_0 \leq 0$, then it is simply not the case that the bond price is at the upper bound.)

Lower bound case: Suppose that $\tilde{q} < \underline{q}$. Then $\psi_0 = 1$ but $\psi_1 < 1$; in fact, we have:

$$\chi\psi_1 B_1 = \chi\frac{M_0}{\underline{q}} - \gamma B$$

Substituting this into the bond flow equations, we get:

$$B_0 = \frac{\alpha B}{\varepsilon + \alpha + \gamma} + \frac{\chi M_0}{(\varepsilon + \alpha + \gamma)\underline{q}} \qquad \text{and} \qquad B_1 = \frac{(\varepsilon + \gamma)B}{\varepsilon + \alpha + \gamma} - \frac{\chi M_0}{(\varepsilon + \alpha + \gamma)\underline{q}}$$

Finally, the flow equation for M_0 in steady state yields, after the usual substitutions:

$$\frac{M_0}{M} = \frac{1}{\varepsilon + \tilde{n}\alpha + \gamma + \chi\left(1 - \frac{1/\underline{q}}{\varepsilon+\alpha+\gamma}\right)} \cdot \left[n_0\gamma + \tilde{n}\alpha + \left(n_0(\underline{q}\gamma - 1) + \frac{\alpha}{\varepsilon + \alpha + \gamma}\right)\frac{B}{M}\right]$$

Clearly, this term can blow up if the term $(\varepsilon + \alpha + \gamma)\underline{q}$ is small enough. What does this mean? It means that if so, then equilibrium is never in the upper bound region where bond sellers are rationed. For any positive bond supply, no matter how large, the equilibrium bond price will be above the lower bound.

However, the formula also clarifies that this is an odd region of the parameter space. $1/\bar{q} \approx (\rho + \gamma)$, so as long as $\varepsilon + \alpha \gg \rho$ then M_0/M is finite. State transitions need only to be so common as to dominate time preference. This is satisfied with a lot of room to spare by any numerical experiment considered in this paper.

Sufficient condition for a stationary equilibrium. So far, we have established that a unique solution to the stationary equations exists. But the costates equations are only valid if every agent is willing to accumulate money; that is, $h/\phi > -T$, so that labor income is enough to cover any tax obligations. (As decumulation makes the distance to zero relevant, it leads to value functions that are nonlinear in money holdings, rendering the model intractable.)

We have already proven that $\mu_0 < \mu_1$, which implies $h_0 < h_1$; thus, the relevant constraint is on state-0 households. By goods market clearing, $\phi = (n_0 h_0 + n_0 h_1)/(\alpha M_1)$; by the government budget constraint in steady state, $T = \gamma M + (\gamma q - 1)B$. Substituting these into $h_0/\phi > -T$, we obtain the equivalent condition (15).

For the sufficient condition involving only exogenous parameters, the hard part is to get a handle on the term M_1/M in steady state. What proportion of the total money stock is held by households in state 1? Consider that $n_1 = \varepsilon/(\varepsilon+\alpha)$ is the proportion of households who are in state 1 at any time; since, however, households in state 1 work harder than those in state 0, and sell bonds in order to get more money, the natural guess is that $M_1/M > n_1$: households in state 1 hold more money on average than those in state 0. (The only way this could go wrong is if state-0 households hold so many bonds that their dividend income advantage overwhelms state-1 households' income advantage from labor and asset trade.) To verify this guess, solve the following block of equations:

$$\begin{aligned} \gamma M_0 &= B_0 + n_0(\alpha M_1 + T) - \varepsilon M_0 - \chi\psi_0 M_0, & M_1 &= M - M_0, \\ \gamma B_0 &= -\varepsilon B_0 + (\alpha + \chi\psi_1)B_1 + \gamma B, & B_1 &= B - B_0. \end{aligned}$$

This is the stationary version of the asset flow equations, except that I have assumed $h_0 = h_1$. Since in actuality $h_0 < h_1$, assuming that labor income is equal for all households biases the result towards a *lower* level of M_1/M. Solving this block yields:

$$\begin{aligned} \frac{M_1}{M} - \frac{\varepsilon}{\varepsilon+\alpha} &= \ldots \\ &\frac{\alpha\left[\varepsilon^2 + (\alpha+\varepsilon)\chi\psi_0\right](\alpha+\gamma+\varepsilon+\chi\psi_0) - (B/M)(\alpha+\varepsilon)(\varepsilon(\gamma+\chi\psi_1) + q\alpha\gamma(\alpha+\gamma+\varepsilon+\chi\psi_1))}{(\alpha+\varepsilon)\left[\alpha^2 + \alpha(\gamma+\varepsilon+\chi\psi_0) + \varepsilon(\gamma+\varepsilon+\chi\psi_1)\right](\alpha+\gamma+\varepsilon+\chi\psi_1)} \end{aligned}$$

Since the denominator is positive, we only need to consider whether the numerator is positive, too. After rearranging, we obtain the equivalent condition:

$$\frac{B}{M} < \frac{\alpha}{\varepsilon+\alpha} \cdot \frac{(\varepsilon+\alpha+\chi\psi_1+\gamma)(\varepsilon^2 + (\varepsilon+\alpha)\chi\psi_0)}{(\varepsilon+\alpha+\chi\psi_1+\gamma)\alpha\gamma q + \varepsilon(\gamma+\chi\psi_1)}$$

The right-hand side is increasing in ψ_0 and decreasing in ψ_1 and q. So the worst-case values are $\psi_0 = 0$, ψ_1, and $q = \overline{q}$ (as defined in Equation (18)). Substituting these, and simplifying, we learn that if:

$$\frac{\varepsilon\alpha}{\varepsilon+\alpha} \cdot \left[\frac{\alpha}{\varepsilon}\gamma\,\overline{q} + \frac{\chi+\gamma}{\varepsilon+\alpha+\chi+\gamma}\right]^{-1},$$

we are assured that $M_1/M > n_1$ in steady state.

Now return to the exact condition (15). The worst-case scenario for the inequality is if

M_1/M is low, but we can substitute it with n_1 to obtain a sufficient condition. Similarly, $h_0 = (v')^{-1}(\mu_0)$ in the numerator cannot be lower than its worst-case value $(v')^{-1}[1/(\rho+\varepsilon+\gamma)/(\rho+\alpha+\gamma)]$ (the solution of μ_0 when $q = \overline{q} = \beta_0/\mu_0$ – a high bond price makes money less valuable, since one of its uses, acquiring bonds, is made harder). On the other hand, the term $n_0 h_0 + n_1 h_1$ cannot be *higher* than $h^{FI} = (v')^{-1}(1)$ – the best-case value. Finally, the worst-case value of the bond price is $\underline{q}$ (as defined in Equation 17). By Proposition 2(d), $\underline{q} < 1/(\rho+\gamma)$, so the fraction $1/(1-\gamma\underline{q})$ cannot blow up.

Applying all of these worst-case inequalities, the first part of the restriction is confirmed:

$$\frac{B}{M} \le \frac{1}{1-\gamma\underline{q}} \left(\gamma + \frac{\varepsilon\alpha}{\varepsilon+\alpha} \cdot \frac{(v')^{-1}\left(\frac{\varepsilon\alpha}{(\rho+\gamma+\varepsilon)(\rho+\gamma+\alpha)}\right)}{(v')^{-1}(1)} \right),$$

together with $n_1 < M_1/M$, is sufficient for condition (15) and thus $h > -\phi T$ for everybody.

If the sufficient condition fails, the underlying restriction $h_0 + \phi T > 0$ must be verified numerically for a candidate equilibrium. This is likely to work more generally, since some of the inequalities used above are quite crude. If, for example, the bond price is above $\underline{q}$ (bonds carry a liquidity premium), then the fraction $1/(1-\gamma q)$ could blow up. In that case, the money earned from placing new bonds on the market is enough to cover the dividend payments, bond issuance pays for itself, and no taxes are needed at all. □

Proof of Proposition 4. A byproduct of the proof of Proposition 3 above. The closed-form expressions for $\underline{q}$ and $\overline{q}$ are derived there.

For the region boundaries, simply take the equilibrium formula for $\tilde{q}$ (Equation A.7), set $\tilde{q} = \underline{q}$ and $\tilde{q} = \overline{q}$ respectively, and invert for B/M. But recall that $\tilde{n}$, denoting the proportion of total output produced by households in state 0, is endogenous and in particular depends on the bond price q, as the formulas make explicit:

$$b_{SR} \equiv (n_0\gamma + \tilde{n}(\underline{q})\alpha) \Big/ \left[n_0 - \frac{\alpha+\chi+\gamma}{\varepsilon+\alpha+\chi+\gamma} + \left(\frac{\varepsilon(\varepsilon+\tilde{n}(\underline{q})\alpha+\chi+\gamma)}{\varepsilon+\alpha+\chi+\gamma} + \gamma\left(n_1 + \frac{\varepsilon+\tilde{n}(\underline{q})\alpha+\gamma}{\chi} \right) \right) \underline{q} \right]$$

$$b_{BR} \equiv (n_0\gamma + \tilde{n}(\overline{q})\alpha) \Big/ \left[n_0 - \frac{\alpha+\chi+\gamma}{\varepsilon+\alpha+\chi+\gamma} + \left(\frac{\varepsilon(\varepsilon+\tilde{n}(\overline{q})\alpha+\chi+\gamma)}{\varepsilon+\alpha+\chi+\gamma} + \gamma\left(n_1 + \frac{\varepsilon+\tilde{n}(\overline{q})\alpha+\gamma}{\chi} \right) \right) \overline{q} \right]$$

□

A.2 Dynamic monetary equilibria

The dynamic form of the Euler equations (A.5) (or indeed 6) is as follows:

$$\rho\mu_{0,t} = \dot{\mu}_{0,t} - \pi_t\mu_{0,t} + \varepsilon[\mu_{1,t} - \mu_{0,t}] + \chi\left[\frac{\beta_{0,t}}{q_t} - \mu_{0,t}\right] \tag{A.8a}$$

$$\rho\mu_{1,t} = \dot{\mu}_{1,t} - \pi_t\mu_{1,t} + \alpha[1 - \mu_{1,t}] \tag{A.8b}$$

$$\rho\beta_{0,t} = \dot{\beta}_{0,t} - \pi_t\beta_{0,t} + \mu_{0,t} + \varepsilon[\beta_{1,t} - \beta_{0,t}] \tag{A.8c}$$

$$\rho\beta_{1,t} = \dot{\beta}_{1,t} - \pi_t\beta_{1,t} + \mu_{1,t} + \alpha[\beta_{0,t} - \beta_{1,t}] + \chi\,[q_t\mu_{1,t} - \beta_{1,t}] \tag{A.8d}$$

The interpretation is the same as before, except that the flow values of assets now also include capital gains terms ($\dot{\mu}_{0,t}$, etc.).

The asset flow equations (14) already include rate of change terms, so we can use them unchanged. The market clearing equations and government budget are also the same as before, taking note of course of the fact that the bond sale S_t does not have to equal γB_t outside of steady state. Similarly, inflation may not equal money growth in every period. Thus, the last object we need to describe is expected inflation $\pi_t = -\dot{\phi}_t/\phi_t$.

Because ϕ is determined by the goods market clearing equation (11), we differentiate that equation with respect to time:

$$\alpha\phi\dot{M}_1 + \alpha\dot{\phi}M_1 = n_0\dot{h}_0 + n_1\dot{h}_1 \tag{A.9}$$

$$\leftrightarrow \qquad \frac{\dot{\phi}}{\phi} + \frac{\dot{M}_1}{M_1} = \frac{n_0\dot{h}_0 + n_1\dot{h}_1}{n_0h_0 + n_1h_1}$$

Next, we differentiate with respect to time Equation (7), for $i = 0, 1$:

$$\dot{h}_i = \frac{1}{v''(h_i)}\dot{\mu}_i \tag{A.10}$$

Finally, we can use Equations (A.8a,b) to substitute for $\dot{\mu}_i$, Equation (14a) to substitute for $\dot{M}_1$, and write:

$$\pi_t \;=\; -\frac{\dot{\phi}}{\phi} \;=\; \underbrace{\frac{\dot{M}_1}{M_1} - \frac{n_0/v''(h_0)\cdot\dot{\mu}_0 + n_1/v''(h_1)\cdot\dot{\mu}_1}{n_0h_0 + n_1h_1}}_{\substack{\text{time derivatives substituted using}\\ \text{(14a) and (A.8a,b)}}} \tag{A.11}$$

With the hard work done, we can now describe a dynamic equilibrium purely in terms of ordinary differential equations, contemporaneous equations, and boundedness conditions.

Definition 2. A dynamic monetary equilibrium is a list of paths $\{h_{0,t}, h_{1,t}, \mu_{0,t}, \mu_{1,t}, \beta_{0,t}, \beta_{1,t}, q_t, \psi_{0,t}, \psi_{1,t}, \pi_t, \phi_t, M_{0,t}, M_{1,t}, B_{0,t}, B_{1,t}, T_t\}$ which satisfy Equations (7), (11), (12), (13), (14), (A.8) and (A.11) for all $t \geq 0$, where the paths of $\{h_{0,t}, h_{1,t}, \mu_{0,t}, \mu_{1,t}, \beta_{0,t}, \beta_{1,t}, q_t, \psi_{0,t}, \psi_{1,t}, \pi_t, \phi_tM_{0,t}, \phi_tM_{1,t}, \phi_tB_{0,t}, \phi_tB_{1,t}, \phi_tT_t\}$ remain bounded, and where $\phi_t \geq 0\ \forall t$. The exogenous variables may be paths as well, provided they are bounded, piecewise continuous, and common knowledge.

A.3 Extension: the asset market is intermediated by dealers

Assume that there is a measure χ of "dealers", special financial firms that intermediate asset trade. Households own a diversified portfolio of dealers' profits. Dealers have the ability to verify and certify bonds (which explains why households are unable to trade bonds directly, including paying for goods with bonds), and they are able to access a frictionless inter-dealer market (so they do not need to hold inventory). At rate χ, households are matched with dealers, and there is no direct asset trade between households. When matched, households and dealers determine the size of the trade and the split of the surplus by Nash bargaining, where the dealer has a bargaining power exponent of $\zeta \in [0, 1]$. (When $\zeta = 0$, the resulting equations are identical to those in the main text.)

First, consider a match between a household with real balances z^M and real bonds z^B, and a dealer with access to the inter-dealer market where the price is q. Denote by s_m the "match-specific" real quantity of bonds sold by the household (bought if negative), and by q_m the match-specific price charged or offered for those bonds. Assume that the value function of money and bonds is again linear (which would need to be verified in a complete analysis), with a marginal value of real balances μ and a marginal value of bonds β. Then the Nash bargaining solution satisfies:

$$(s_m, q_m) \in \arg\max_{s_m, q_m} \left\{ (\mu q_m s_m - \beta s_m)^{1-\zeta} (-q_m s_m + q s_m)^{\zeta} \right\}$$
$$\text{subject to } s_m \in [-z^M/q_m, z^B]$$

The bargaining solution has two cases, corresponding to whether the household buys or sells bonds. The first case, where the household is a buyer, obtains if $\beta/\mu > q$:

$$s_m = -\frac{z^M}{q_m} \quad \text{and} \quad q_m = \left[\zeta \left(\frac{\beta}{\mu} \right)^{-1} + (1-\zeta) q^{-1} \right]^{-1}$$

So the household will spend all of its money to buy bonds, and the ask price charged is the *harmonic mean* of the parties' marginal rates of substitution, weighted by their bargaining power. The second case, where the household is a seller, obtains if $\beta/\mu < q$:

$$s_m = z^B \quad \text{and} \quad q_m = \zeta \frac{\beta}{\mu} + (1-\zeta) q$$

So the household will sell all of its bonds, and the bid price offered is the *arithmetic mean* of the parties' marginal rates of substitution, weighted by their bargaining power.

In equilibrium, because $\beta_0/\mu_0 > \beta_1/\mu_1$, households in state 0 will buy bonds at the ask price and households in state 1 will sell bonds at the bid price. Substituting the results into the households' Euler equations, these equations take the following form:

$$\rho\mu_0 = \dot{\mu}_0 - \pi\mu_0 + \varepsilon[\mu_1 - \mu_0] + \chi(1-\zeta)\left[\frac{\beta_0}{q} - \mu_0\right] \tag{A.12a}$$

$$\rho\mu_1 = \dot{\mu}_1 - \pi\mu_1 + \alpha[1 - \mu_1] \tag{A.12b}$$

$$\rho\beta_0 = \dot{\beta}_0 - \pi\beta_0 + \mu_0 + \varepsilon[\beta_1 - \beta_0] \tag{A.12c}$$

$$\rho\beta_1 = \dot{\beta}_1 - \pi\beta_1 + \mu_1 + \alpha[\beta_0 - \beta_1] + \chi(1-\zeta)\left[q\mu_1 - \beta_1\right] \tag{A.12d}$$

We can see that as far as households' decision making is concerned, giving dealers bargaining power $\zeta > 0$ is equivalent to increasing the trading delay (confirming the result of Lagos and Rocheteau, 2009). General equilibrium is a little bit more complicated now because dealer's profits need to be remitted to households, but the extension is straightforward and not much happens in terms of results. Since the main purpose of the model is understanding monetary intervention in illiquid markets, including dealers is not essential. It may matter for empirical purposes, however, or for extensions that take financial market microstructure more seriously.

A.4 Extension: bonds have direct liquidity

In the model from the main text, money is the only means of payment in the goods market; however, bonds can be traded for money on a market, which imbues them with *indirect liquidity* properties Geromichalos and Herrenbrueck (2016a). Alternatively, or in addition, we can consider the *direct liquidity* paradigm (Geromichalos et al., 2007; Lagos and Rocheteau, 2008; Lester et al., 2011): bonds also serve as means of payment, but only with probability $\eta \in [0, 1]$, and whether or not is revealed at the point of the purchase. (So far, only one paper considers both paradigms together (Geromichalos et al., 2019); it finds that assets other than money can be both indirectly and directly liquid, but assets which trade on more liquid secondary markets are less likely to be accepted as media of exchange.)

Here, in the continuous-time environment with household heterogeneity, giving bonds direct liquidity properties is awkward and raises technical issues that distract from the research questions. Number one, can tax obligations be paid for with bonds? Number two, in the indirect liquidity model money is valued for any inflation rate (given that $v'(0) = 0$), so there is no need to keep track of an upper bound for inflation. This is not the case for the direct liquidity model, even if bonds are an inferior means of payment. Number three, the most serious issue, is the following.

With regards to bond trade in the asset market, there are *two* household types with distinct marginal rates of substitution (state-0 and state-1, β_0/μ_0 versus β_1/μ_1), so there is a natural direction of trade and the bond price must be between the two MRSs. However, in the goods market, there are *three* types of participants: workers in state 0, workers in state 1, and imminent consumers (transitioning from state 1 to state 0). Thus, there are three relevant

MRSs when money is valued: β_0/μ_0 and β_1/μ_1 for workers in either state, and β_0 for imminent consumers (since their marginal valuation of real balances is 1 by construction). Plus, in case money is not valued, β_1 (the marginal value of bonds to state-1 workers). Since β_0 and β_1/μ_1 cannot be unambiguously ranked, and since the market price of bonds in the goods market must be between β_0/μ_0 (upper bound) and whichever of $(\beta_0, \beta_1/\mu_1)$ is larger (lower bound), but could be interior or at a corner, and since money might or might not be valued, the goods market clearing solution has five possible branches – excluding knife-edge cases. Denoting the amount of goods that one bond will buy by $\hat{q}$, these are:

- $\hat{q} = \beta_0/\mu_0$. State-0 workers accept both bonds and money as income, state-1 workers only take money
- $\beta_0/\mu_0 > \hat{q} > \max\{\beta_1/\mu_1, \beta_0\}$. State-0 workers accept only bonds, state-1 workers accept only money, and consumers spend all their money and (if allowed) bonds
- $\beta_0/\mu_0 > \hat{q} = \beta_0 > \beta_1/\mu_1$. State-0 workers accept only bonds, state-1 workers accept only money, and consumers spend all their money but only some of their bonds
- $\beta_0/\mu_0 > \hat{q} = \beta_1/\mu_1 > \beta_0$. All workers accept bonds, only state-1 workers accept money, and consumers spend all their money and (if allowed) bonds
- $\mu_0 = \mu_1 = 0$ (money is not valued), and all workers accept bonds at price $\hat{q} > \beta_1 > \beta_0$

Due to this complexity, a full general equilibrium analysis of the direct liquidity case is beyond the scope of this paper. Nevertheless, there are some things we can say for sure. First, we can show that trade in the frictional asset market must still flow in the natural direction even if bonds can be used as payment, as long as money is valued:

Proposition A.1. *Let $\eta < 1$, and money is valued ($\mu_0, \mu_1 > 0$). Then $\beta_0/\mu_0 > \beta_1/\mu_1$.*

Proof. For simplicity, assume that $\chi \approx 0$, so asset trade is rare. (The extension to $\chi > 0$ is a straightforward adaptation of the proof of Proposition 2(d) in Appendix A.1.) Then, the steady-state Euler equations are as follows:

$$(\rho+\gamma)\mu_0 = \varepsilon(\mu_1 - \mu_0), \quad (\rho+\gamma)\beta_0 = \mu_0 + \varepsilon(\beta_1 - \beta_0),$$
$$(\rho+\gamma)\mu_1 = \alpha(1 - \mu_1), \quad (\rho+\gamma)\beta_1 = \mu_1 + \alpha(1-\eta)(\beta_0 - \beta_1) + \alpha\eta\left(\max\left\{\hat{q}, \beta_0\right\} - \beta_1\right),$$

where the only difference from the main text is that bonds can be used to pay for goods with probability η, in which case one bond buys $\hat{q}$ of goods. Focus on the case where $\hat{q} > \beta_0$, otherwise consumers would simply hold on to their bonds voluntarily and direct bond liquidity becomes irrelevant.

We can divide the equations by μ_0 and μ_1, respectively, and rearrange them to get:

$$\mu_0 = \frac{\varepsilon}{\rho+\gamma+\varepsilon}\,\mu_1 \qquad (\rho+\gamma)\frac{\beta_0}{\mu_0} = 1 + \varepsilon\,\frac{\mu_1}{\mu_0}\cdot\frac{\beta_1}{\mu_1} - \varepsilon\,\frac{\beta_0}{\mu_0}$$

$$\mu_1 = \frac{\alpha}{\rho+\gamma+\alpha} \qquad\qquad (\rho+\gamma)\frac{\beta_1}{\mu_1} = 1 + \alpha\eta\,\frac{\hat{q}}{\mu_1} + \alpha(1-\eta)\,\frac{\mu_0}{\mu_1}\cdot\frac{\beta_0}{\mu_0} - \alpha\,\frac{\beta_1}{\mu_1}$$

Plug in for the fractions μ_1/μ_0 and $\hat{q}/\mu_1$, and rearrange the latter two equations to become:

$$(\rho+\gamma+\varepsilon)\frac{\beta_0}{\mu_0} = 1 + (\rho+\gamma+\varepsilon)\frac{\beta_1}{\mu_1} \tag{A.13}$$

$$(\rho+\gamma+\alpha)\frac{\beta_1}{\mu_1} = 1 + \eta(\rho+\gamma+\alpha)\hat{q} + (1-\eta)\frac{\varepsilon(\rho+\gamma+\alpha)}{\rho+\gamma+\varepsilon}\cdot\frac{\beta_0}{\mu_0} \tag{A.14}$$

Subtracting Equation (A.14) from (A.13):

$$(\rho+\gamma+\varepsilon)\frac{\beta_0}{\mu_0} = (2\rho+2\gamma+\varepsilon+\alpha)\frac{\beta_1}{\mu_1} - \eta(\rho+\gamma+\alpha)\hat{q} - (1-\eta)\frac{\varepsilon(\rho+\gamma+\alpha)}{\rho+\gamma+\varepsilon}\cdot\frac{\beta_0}{\mu_0}$$

Adding $(\rho+\gamma+\alpha)\beta_0/\mu_0$ to both sides, and rearranging:

$$\begin{aligned}
(2\rho+2\gamma+\varepsilon+\alpha)\frac{\beta_0}{\mu_0} &= (2\rho+2\gamma+\varepsilon+\alpha)\frac{\beta_1}{\mu_1} - \eta(\rho+\gamma+\alpha)\hat{q} \quad \dots \\
&\quad + \left(\rho+\alpha-(1-\eta)\frac{\varepsilon(\rho+\gamma+\alpha)}{\rho+\gamma+\varepsilon}\right)\frac{\beta_0}{\mu_0} \\
\leftrightarrow \qquad \frac{\beta_0}{\mu_0} &= \frac{\beta_1}{\mu_1} + \frac{\rho+\gamma+\alpha}{2\rho+2\gamma+\varepsilon+\alpha}\left(\frac{\rho+\gamma+\eta\,\varepsilon}{\rho+\gamma+\varepsilon}\cdot\frac{\beta_0}{\mu_0} - \eta\,\hat{q}\right)
\end{aligned}$$

Since $\beta_0/\mu_0 \geq \hat{q}$ (otherwise no worker would accept bonds in exchange for goods) and $(\rho+\gamma+\eta\,\varepsilon)/(\rho+\gamma+\varepsilon) > \eta$, the term in parentheses is positive. This proves the claim. □

Thus, trade in the bond market always flows in the natural direction: money is directed into the hands of those who need it more. Since most of the results of the paper can be traced back to this fact, they do not depend on the assumption that money is the only medium of exchange – they only require that money is the *best* medium of exchange.

Another thing we can do with the direct liquidity extension is to analyze how the steady-state asset valuations would change for the three special cases of the goods-market bond price $\hat{q}$. Including asset trade, the steady-state Euler equations are as follows:

$$\begin{aligned}
(\rho+\gamma)\mu_0 &= \varepsilon(\mu_1-\mu_0) + \chi\left(\frac{\beta_0}{q} - \mu_0\right) \\
(\rho+\gamma)\mu_1 &= \alpha(1-\mu_1) \\
(\rho+\gamma)\beta_0 &= \mu_0 + \varepsilon(\beta_1-\beta_0) \\
(\rho+\gamma)\beta_1 &= \mu_1 + \alpha(1-\eta)(\beta_0-\beta_1) + \alpha\eta(\hat{q}-\beta_1) + \chi\,(q\,\mu_1-\beta_1)
\end{aligned}$$

Accordingly:

- Let $\hat{q} = \beta_0$, which is obtained if ηB_1 is large (consumers hold more bonds than they can spend). Clearly, the Euler equations are exactly the same as in the main text.
- Let $\hat{q} = \beta_1/\mu_1 > \beta_0$, which is obtained if ηB_1 is small (consumers spend all their bonds) but not too small ($\alpha\eta B_1 > n_0 h_0$, so state-0 workers do not take money and even state-1 workers take some bonds as income). In this case, the flow real bond value $(\rho+\gamma)\beta_1$ gets a small boost, equal to $\alpha\eta(\beta_1/\mu_1 - \beta_0)$.
- Let $\hat{q} = \beta_0/\mu_0$, which is obtained if $\alpha\eta B_1 < n_0 h_0$ (even state-0 workers take some money as income). In this case, the flow real bond value $(\rho+\gamma)\beta_1$ gets a larger boost, equal to $\alpha\eta(1-\mu_0)\beta_0/\mu_0$. However, this boost will never be big enough to reverse the direction of asset trade, as shown in Proposition A.1 above.

Since asset trade still moves bonds away from state-1 households and money towards them, consumers in this model will actually not have that many bonds to use as payment instruments as long as $\chi\psi_1$ is large enough. Thus, it stands to reason that the general equilibrium results are not strongly affected by a small amount of direct liquidity ($\eta > 0$ but not large) as long as bonds are scarce enough to be priced at a premium (which is the interesting case anyway) and χ is large enough (which is plausible).

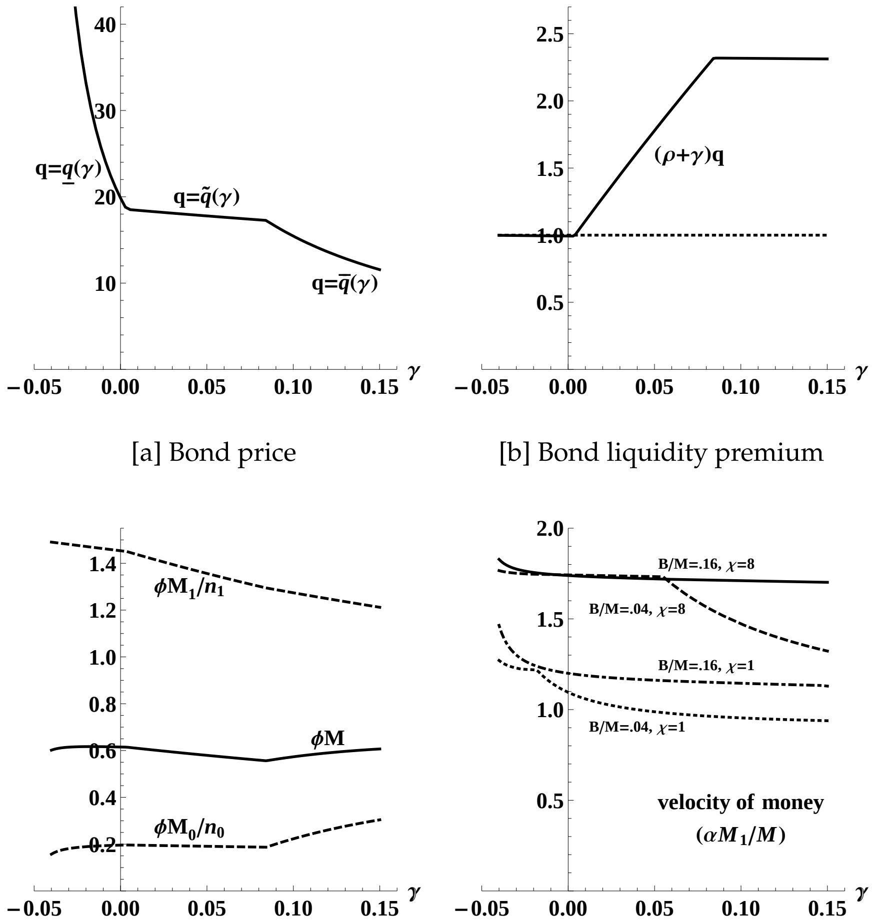
A.5 Comparative statics of inflation and trading frequency

The discussion in Section 4.1 has already suggested that the comparative statics of inflation are important, not least because they are entwined with the comparative statics of bond supply under fiscal dominance. Focus on the baseline model with monetary dominance, and begin with characterizing the bond price bounds $\underline{q}$ and $\overline{q}$ (Equations 17-18).

Proposition A.2. *For a given set of parameters $(\rho, \varepsilon, \alpha, \chi)$, the bond price bounds satisfy:*

1. *If $\gamma > -\rho$ (away from the Friedman rule), then $\underline{q} < 1/(\rho+\gamma) < \overline{q}$;*
2. *If $\gamma > -\rho$ (away from the Friedman rule), then the upper-bound liquidity premium (i.e., ratio of bond price and fundamental value) $(\rho+\gamma)\overline{q}$ is increasing in inflation if and only if $\varepsilon+\alpha > \chi$;*
3. *As $\gamma \to -\rho$ (near the Friedman rule), the ratios of bond price and fundamental value converge to: $(\rho+\gamma)\underline{q} \to 1$ and $(\rho+\gamma)\overline{q} \to 1 + \chi/(\varepsilon+\alpha)$;*
4. *As $\gamma \to \infty$, the ratios of bond price and fundamental value converge to: $(\rho+\gamma)\underline{q} \to 1$ and $(\rho+\gamma)\overline{q} \to 2$.*

Proof. For the closed-form solution of $\underline{q}$, see Equation (17); the fact that $\underline{q} < 1/(\rho+\gamma)$, reflecting an illiquidity discount, is inequality (10) in Proposition 2(d). The limits of low and high inflation follow from some tedious algebra.



[a] Bond price

[b] Bond liquidity premium

[c] Real balances per household: state-0, state-1, and average

[d] Velocity of money (varying parameters)

Figure A.1: Comparative statics of inflation. Total output is proportional to ϕM_1 (panel [c]). Key parameters: $\rho = .05$, $\varepsilon = 1$, $\alpha = 2$, $\chi = 4$, $B/M = .08$, except as indicated in panel [d].

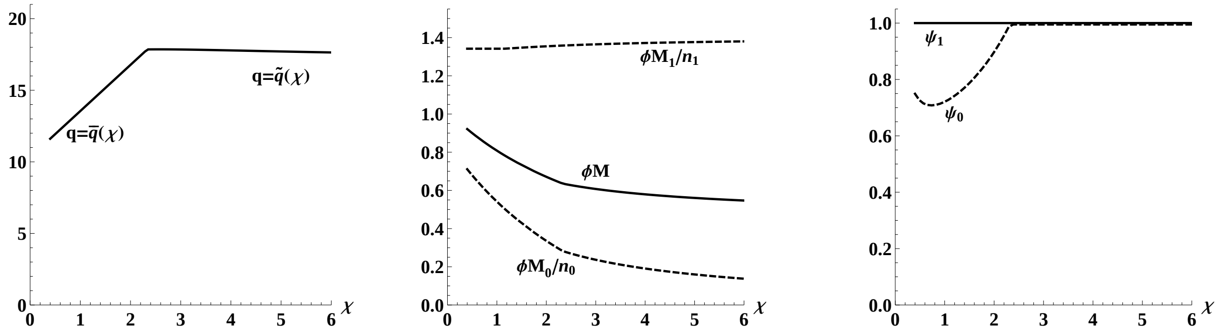
And via Equation (18):

$$\overline{q} = \frac{\beta_0}{\mu_0} = \frac{1}{\rho+\gamma} \cdot \left[1 + \frac{\rho+\gamma+\chi}{\rho+\gamma+\varepsilon+\alpha}\right]$$

Clearly, $\overline{q} > 1/(\rho+\gamma)$, and the derivative of the liquidity premium with respect to γ is:

$$\frac{d}{d\gamma}[(\rho+\gamma)\overline{q}] = \frac{\varepsilon+\alpha-\chi}{(\rho+\gamma+\varepsilon+\alpha)^2}$$

And the limits of low and high inflation follow directly from the formula. $\square$



[a] Bond price

[b] Real balances per household: state-0, state-1, and overall

[c] Trading probabilities (ψ_0 offset for visibility)

Figure A.2: Comparative statics of trading frequency. Total output is proportional to ϕM_1 (panel [b]). Key parameters: $\rho = \gamma = .05$, $\varepsilon = 1$, $\alpha = 2$.

The fundamental value of a nominal consol bond is $1/(\rho + \gamma)$, obviously decreasing in inflation. However, the liquidity premium $(\rho + \gamma)q$, the ratio of the bond price to its fundamental value, can be non-monotonic in inflation. As Figure 6 shows, for low enough inflation the bond price is always at its lower bound, and the liquidity premium must thus be less than one. (An illiquidity discount rather than a liquidity premium.) For high enough inflation, the bond price must be at the upper bound, and the previous proposition showed that the liquidity premium is above 2 but decreasing in inflation whenever $\varepsilon + \alpha > \chi$. In the interior, it is M_0/B_1, the ratio of bond demand to supply, that determines whether the price is at a boundary or in the interior; this term could be increasing in inflation, and certainly the liquidity premium must be increasing. Figure A.1 shows these results, along with the effects of inflation on real balances and the velocity of money.

The comparative statics of bond liquidity, represented by the rate of matching in the decentralized asset market χ, on the bond price bounds $(\underline{q}, \overline{q})$ are illustrated in Figure 6 in the main text. Further comparative statics results are illustrated in Figure A.2. Of note is the fact that in this model, higher bond tradability tends to increase the velocity of money $(\alpha M_1/M)$ and thereby output and welfare, as it concentrates money in the hands of state-1 households and bonds in the hands of state-0 households, thus allocating the assets efficiently in the hands of those who value them the most. This is in contrast to other models of illiquid bonds (e.g., Kocherlakota (2003), Berentsen, Huber, and Marchesiani, 2014, Geromichalos and Herrenbrueck, 2016a,b, and Herrenbrueck and Geromichalos, 2017), where higher bond tradability can be bad for welfare.