Quantitative Easing, Frictional Asset Markets, and the Liquidity Channel of Monetary Policy

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ABSTRACT

How do central bank purchases of illiquid assets affect interest rates and the real economy? To answer this question, I construct a flexible model of asset liquidity with heterogeneous households – some households need money more urgently than others and hold more of it. Households (and the government) can trade in asset markets, but these are subject to frictions. I find that open market purchases of illiquid assets are fundamentally different from helicopter drops: asset purchases stimulate private demand for consumption goods at the expense of demand for assets and investment goods, while helicopter drops do the reverse. A temporary program of quantitative easing can therefore cause a 'hangover' of elevated yields and depressed investment. When assets are already scarce, further purchases can crowd out the private flow of funds and cause high real yields and disinflation – a liquidity trap. In the long term, lowering the stock of government debt reduces the supply of liquidity but increases the capital-output ratio, so the ultimate impact on output is ambiguous.

JEL Codes: E31, E40, E50, G12

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1 Introduction

The recent quantitative easing programs in Japan, the United States, and the Eurozone have renewed theoretical interest in the question of how monetary policy can affect long-term interest rates, borrowing costs, and the real economy. With short-term rates near zero, central banks hoped to gain traction with purchases of illiquid assets, such as long-term government bonds, federal agency debt, and privately-issued mortgage-backed securities. The empirical literature on quantitative easing suggests that the programs were effective in reducing yields, but due to the lack of a suitable counterfactual, measuring the effects on the broader economy is difficult. Consequently, understanding how the prices of illiquid assets relate to each other and to the quantities in supply and demand, and how government intervention can affect these relationships, remains a priority for macroeconomic theory.

For this purpose, I construct a general equilibrium model with multiple assets – money, long-term bonds, and physical capital – that differ in liquidity. Households receive random opportunities to purchase goods with money, and they are heterogeneous in how soon they expect these opportunities to arrive. Thus, some households value money more than others, giving them a motive to trade in financial markets. Bonds and capital are illiquid in the sense that they cannot be used as media of exchange, and furthermore trade in financial markets is subject to frictions. However, as long as these assets can be traded at all, they inherit 'moneyness' from the fact that people who anticipate needing money in the future know that they can *liquidate* their other assets when the need arises. As a consequence, and in contrast to standard asset pricing theory, bonds and capital are not only valued for their dividend streams, but also for how easily they can be liquidated, and at what price. Capital, in particular, has a *dual role*: it is an imperfect substitute to money and bonds as a store of value, and it is a productive input. Thus, capital links the financial and real sides of the economy, giving scope to monetary policy to affect the economy even in the absence of nominal rigidities.

The chief result of the paper is thus that household heterogeneity and asset market frictions combine to establish a liquidity channel of monetary policy, which sometimes supports and sometimes competes with the standard investment channel (lower interest rates promote investment). This liquidity channel has three components: an *immediate* one, a *short-term* one, and a *permanent* one. First, the demand curves of bonds and capital are downward sloping, therefore a direct intervention will immediately affect their prices. Second, household heterogeneity gives rise to a short-term portfolio effect. People who plan to buy goods need money more than other assets, therefore they are the natural sellers of assets. As open-market purchases by their nature direct cash towards asset sellers, they stimulate the demand for goods at the expense of the demand for assets (net of the purchases themselves) along the transition to the new steady state. Third, because bonds perform a useful service in this economy – they help direct money into the hands of those who need it the most – the aggregate supply of bonds also has a portfolio effect on asset prices, investment, and the value and velocity of money in steady state.

This result implies several lessons for monetary policy. First, contrast an open-market purchase of illiquid assets with a direct transfer of money to everyone: a "helicopter drop". The asset purchase activates all three components of the liquidity channel; the first one while the intervention is ongoing, the second one in the short-term aftermath, and the third one persisting in steady state. By contrast, the helicopter drop only activates the second, short-term component, and in the opposite direction of the asset purchase: it directs cash *away from* people seeking to spend, and stimulates the demand for assets at the expense of the demand for goods. Thus, while helicopter drops and open-market purchases are both "money injections" that increase the long-run price level, they have fundamentally different effects on financial markets.

The second lesson is that while a purchase of illiquid bonds generally has the expected effects – higher bond prices due to scarcity, and at least temporary inflation due to more money in circulation – there is an important exception. If bonds are so scarce that the marginal bond buyer is rationed (a liquidity trap), and if the fiscal deficit is fixed in real terms, then the bond purchase may result in *lower bond prices and lower inflation* in the long run. The reason for this surprising result is that the velocity of money is endogenous and increasing in the supply of bonds. Removing bonds from the market results in potential buyers holding on to their money for longer, a textbook increase in "money demand" that defeats the intent of the expansion of money supply.

What does this mean for the prospects of a quantitative easing program – a large-scale purchase of real, long-term, and illiquid assets? Both while the intervention is ongoing and in the long run, such purchases will tend to reduce the yields on the purchased assets and can thereby stimulate investment and output. In this sense, quantitative easing "works". However, the model also reveals several important caveats. First, the intervention reallocates money from asset buyers towards asset sellers; just as this stimulates private demand for goods in the short run, it crowds out private demand for assets, which can cause a hangover of high yields and depressed investment in the short term after the intervention ends. Second, assets such as government bonds perform a useful function in this economy, so reducing their supply causes a direct efficiency loss that must be weighed against the indirect effect on investment. A calibration of the model suggests that this concern is realistic. Third, when illiquid assets are already scarce enough to push the economy into a liquidity trap, further purchases only slow down the circulation of money. This may result in disinflation and higher real interest rates, defeating the intent of the intervention.

The argument that frictions in portfolio management are the source of monetary non-neutrality and make intervention effective has a long tradition in monetary theory (Baumol, 1952; Tobin, 1956), continued by Alvarez and Lippi (2009, 2013). (These papers have in common with mine that households' asset portfolios are heterogeneous and can be manipulated by monetary policy, but they do not include a capital asset that links monetary policy and investment.) One offshoot of this approach is the "limited participation" literature, in which not all agents participate in asset markets, and some agents face cash-in-advance or borrowing constraints (Fuerst, 1992; Alvarez, Atkeson, and Kehoe, 2002; Williamson, 2006). One closely related paper to mine is by Alvarez and Lippi (2014), who also study the interaction of asset market frictions, portfolio heterogeneity, and monetary policy, and derive some similar results (e.g. the short-term liquidity effect of money on interest rates). There are two main differences. First, in Alvarez and Lippi (2014), real balances enter the utility function (and key results are derived from the shape of the utility function) whereas in my paper, money has value because it is the medium of exchange (and key results are derived from the properties of the asset markets). Second Alvarez and Lippi (2014) analyze an endowment economy and focus on empirical correlations between financial variables, whereas my focus is on the effects of monetary policy on investment and output.

My model is a hybrid of a monetary-search model in the tradition of Lagos and Wright (2005) and a model of frictional asset markets in the tradition of Duffie, Gârleanu, and Pedersen (2005), Lagos and Rocheteau (2009), and Trejos and Wright (2016).[1] The literature which uses search theory to study monetary policy and asset prices is extensive; Geromichalos, Licari, and Suárez-Lledó (2007), Berentsen, Camera, and Waller (2007), Lagos (2010, 2011), Berentsen and Waller (2011), and Rocheteau and Wright (2012) are prominent milestones. Here, I study the pricing of an asset that cannot be used in exchange but has endogenous liquidity properties because it can be traded for money in a frictional asset market, as do Geromichalos and Herrenbrueck (2016), Lagos and Zhang (2015), Berentsen, Huber, and Marchesiani (2014), Mattesini and Nosal (2016), Huber and Kim (2017), and Herrenbrueck and Geromichalos (2017), who dubbed this property "indirect liquidity".[2]

The effect of monetary policy on the accumulation of physical capital has been the focus of a literature going back to Tobin (1965), with recent contributions by Lagos and Rocheteau (2008), He and Krishnamurthy (2013), Rocheteau and Rodriguez-Lopez (2014), and Venkateswaran and Wright (2014). In these works, anticipated inflation leads to a higher capital stock, often inefficiently high. However, Aruoba, Waller, and Wright (2011) generate

[1] Nosal and Rocheteau (2011) and Lagos, Rocheteau, and Wright (2017) provide excellent surveys.

[2] Applying models of asset market frictions to markets for government bonds is sometimes challenged because these markets are considered highly liquid. However, the frictional model is valid as long as the markets are not *perfectly* liquid – of course no real-world market is – and the rest is comparative statics. For example, Ashcraft and Duffie (2007) documented the relevance of frictions in the federal funds market, which at the time was one of the most liquid markets in existence. Vayanos and Weill (2008) and Andreasen, Christensen, Cook, and Riddell (2016) report yield spreads of 30-60 basis points between different classes of Treasuries that only differ in their liquidity. Finally, if Treasuries were exactly as liquid as cash then they could not be priced at a positive nominal yield by agents who also hold money. But they are.

an opposing result: because the inflation tax hurts production, it also reduces investment in productive inputs: in particular, capital. My model nests both outcomes. If capital is scarce relative to other assets, then people value it highly for its liquidity properties, and if in addition the labor supply is inelastic, then moderate inflation increases capital accumulation and output. If, on the other hand, capital is so abundant that (at the margin) it is not valued for liquidity, and instead the labor supply is elastic with respect to the marginal value of income, then inflation reduces both the capital stock and output.

Finally, in its emphasis on household heterogeneity as a driver of asset trade, this paper also continues a tradition going back to Bewley (1980, 1983) and Scheinkman and Weiss (1986). A recent contribution by Cúrdia and Woodford (2011) has in common with my paper that households are heterogeneous and differ in their demand for liquidity; they model it as patience shocks that make some households want to borrow, whereas I model it as random differences in how soon households expect to need money.

The paper is organized as follows. Section 2 develops the baseline model with a focus on frictional asset markets. In Section 3, this model is used to analyze monetary neutrality, the short-run and long-run liquidity channel, and the liquidity trap. In Section 4, the model is extended to include physical capital with a dual role: it serves as an input to production and as a saving vehicle traded in asset markets. This establishes a link between monetary policy and investment, which is analyzed in Section 5. Section 6 discusses the theoretical implications for quantitative easing policies, and Section 7 concludes. The Appendix contains proofs and further details, and a calibration is provided in a Web Appendix.

2 A model of frictional asset markets

The monetary environment and the structure of goods and labor markets are similar to Rocheteau, Weill, and Wong (2015). There are three innovations. First, households are heterogeneous in how soon they expect to need money. Second, there are financial assets in addition to money. Third, these assets can be traded in frictional asset markets à la Duffie, Gârleanu, and Pedersen (2005) and Geromichalos and Herrenbrueck (2016). Because households are heterogeneous, there exist gains from trade: some households would like to sell assets for money, others would like to buy them.

This section focuses on the simplest version of the model, with money and government bonds but no physical capital. The role of physical capital will be explored in Section 4.

2.1 Environment

Time $t = [0, \infty)$ is continuous, measured in "years" for concreteness. (The household's problem will be formally solved in discrete time and analyzed in the continuous-time limit: see

Appendix A.1.) There are two types of agents: households and a government. Households have unit measure and are infinitely lived. The government is a single consolidated authority that can create assets, make transfers, and collect taxes.

There are four commodities in the economy. The first is a lumpy consumption good, denoted by c, which can only be consumed as a stock at certain random opportunities. It will serve as the numéraire in this economy. The second is labor effort, denoted by h, which is expended as a flow. Both of these commodities are perishable and generate utility. The other two commodities are assets: they are perfectly durable and do not generate utility. Fiat money, denoted by m, pays no dividend; by contrast, the final commodity is a real consol bond b, which pays a flow dividend of one unit of numéraire per year (and never matures). Both money and bonds are perfectly divisible.

The supply side of the economy is easily described. Each household owns $\bar{h} < \infty$ units of labor. By working, a household can transform labor h into the good c at a constant marginal cost of 1. The supplies of bonds and money, $B(t)$ and $M(t)$, are controlled by the government.

Households are ex-ante identical but cycle through a two-state Markov process, where the states are called 0 and 1. In state 1, households receive opportunities to consume the good c at Poisson arrival rate $\alpha > 0$. Immediately after such shocks, they transition back into state 0, an assumption which is made without loss of generality.[3] While in state 0, households never receive such opportunities, but they randomly transition to state 1 at Poisson arrival rate $\varepsilon > 0$. Hence, the process represents *information*: state 1 represents a 'high likelihood to consume' (because it takes $1/\alpha$ years on average for the consumption shock to arrive), while state 0 represents a 'low likelihood to consume' (because it takes $1/\alpha + 1/\varepsilon$ years on average for *both* shocks to arrive). Figure 1 provides an illustration.

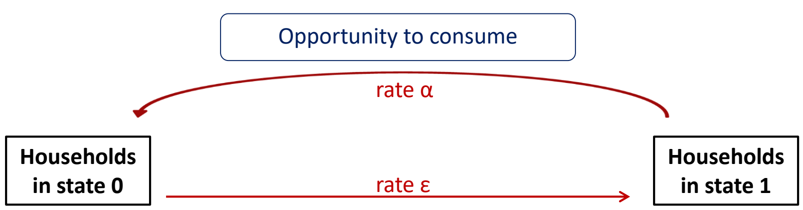


Figure 1: Households cycle through a two-state Markov process

These shocks could be interpreted in two ways. First, the consumption need may be a taste shock that arrives in two stages; one stage provides warning ("I may be hungry soon"), and the other provides confirmation ("I am hungry now"). An alternative interpretation is

[3] Getting ahead of the story: the value functions will be linear and the marginal rates of substitution between money and other assets will not depend on a household's portfolio, merely on its state. Households in state 1 are the only ones with a chance to use money, therefore they will value it more. If they expected to stay in state 1 after making the purchase, they would still value money more than households in state 0.

that one shock represents desire to consume ("I am hungry"), but the market for consumption goods is subject to search-and-matching frictions, so it takes time to find the right seller ("I found a place to eat"). For concreteness, I will use the latter interpretation throughout the rest of the paper. Since it will turn out that consumption needs to be paid for with a liquid asset, we can also interpret state 0 as characterized by a low demand for liquidity, and state 1 by a high demand for liquidity.

Let n_i denote the measure of households in state $i = 0, 1$. In steady state, we must have $n_0 = 1 - n_1 = \alpha/(\varepsilon + \alpha)$, and to make things simpler, we maintain for the rest of the paper the assumption that this relationship holds at all times.

Households discount time at rate $\rho > 0$. Labor effort h generates flow disutility $v(h)$, and consumption of c units of the consumption good (at random time T_1) generates utility c: thus, marginal utility is constant and normalized to 1. As a result, we can write the expected utilities of a household in state 0, $U_0(t)$, and in state 1, $U_1(t)$, in the recursive form:

$$U_0(t) = \mathbb{E}\left\{-\int_t^{T_0} \left[e^{-\rho(\tau-t)}\, v(h(\tau))\right] d\tau + e^{-\rho(T_0-t)}\, U_1(T_0)\right\}$$
$$U_1(t) = \mathbb{E}\left\{-\int_t^{T_1} \left[e^{-\rho(\tau-t)}\, v(h(\tau))\right] d\tau + e^{-\rho(T_1-t)}\, [c(T_1) + U_0(T_1)]\right\}$$

where the first expectation is over the random time T_0 (which arrives at rate ε), and the second expectation is over the random time T_1 (which arrives at rate α).

The function v is strictly increasing and convex. Furthermore, assume $v'(0) = 0$ and $v'(\bar{h}) > 1$; given that the marginal utility of consumption is 1, this implies that in the full-insurance benchmark and in any equilibrium, the constraint $h \leq \bar{h}$ will never bind.

There are two markets, illustrated in Figure 2: a goods market and an asset market. Households are anonymous and take prices as given in all markets. In the goods market, all households use their labor to produce a flow of goods, while only those households who are hit with the α-shock demand of goods. This gives rise to a quantity mismatch: at each point in time, a positive measure of workers is producing an infinitesimal quantity of goods (that is, a flow), while an infinitesimal measure of consumers wants to consume a positive quantity of goods (a stock). Because of this mismatch, and because credit is not feasible due to anonymity, a medium of exchange is necessary. For the purpose of this paper, I assume that only money can be recognized by everyone; households who own bonds know what their own bonds look like, but they cannot verify the authenticity of the bonds that a prospective buyer holds. Hence, money is the only feasible means of payment.[4]

The rate of transformation of labor into goods is 1, thus the real wage is 1 at all times.

[4] Nosal and Wallace (2007), Rocheteau (2009, 2011), Li and Rocheteau (2011), and Lester, Postlewaite, and Wright (2011) analyze in more depth how money emerges as the medium of exchange. And in Appendix A.4, I consider an extension where bonds can also be used as a means of payment, but with a probability less than one.

Denote the price of money in terms of output by ϕ, and express any money holdings m as real balances $z \equiv \phi m$, so that we can express equilibrium in terms of stationary variables. Correspondingly, the inflation rate is defined to be $\pi \equiv -\dot{\phi}/\phi$.

Access to the asset market is subject to a friction: households in either state enter the market randomly at Poisson rate $\chi > 0$. When in the market, households can offer to sell as many bonds as they own, or ask to buy as many bonds as they have money to afford, at real price q (measured in terms of real balances per bond). As in the goods market, pricing is competitive.[5]

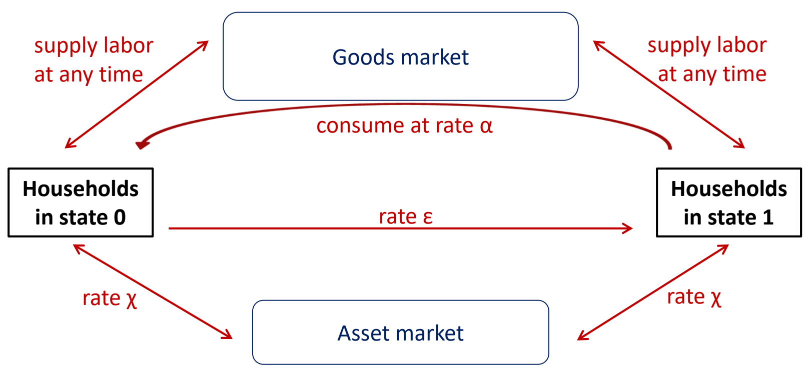


Figure 2: Illustration of the market structure sans government intervention

The government can make lump-sum transfers T of real balances to households (or collect taxes if $T < 0$). They are lump-sum in terms of applying equally to all households, but it is important to keep in mind that they are being assessed as *flows*: they affect the rate of change of households' money holdings, not the holdings directly. The government also has to service its debt by paying a flow dividend of one unit of real balances to the owner of a bond. The government can sell new bonds at flow rate $S \in \mathbb{R}$ in the decentralized bond market (or buy them back if S is negative).

2.2 The full-insurance benchmark

Suppose that all households pool together their labor efforts to insure themselves against the consumption shock. They maximize welfare, subject to the aggregate resource constraint:

$$\max \left\{ \int_0^\infty e^{-\rho t} \Big(-v[h(t)] + \alpha n_1 c(t) \Big) dt \right\} \quad \text{subject to:} \quad \alpha n_1 c(t) = h(t) \quad \text{and} \quad h_t \leq \bar{h},$$

[5] One may ask why households can recognize the bonds owned by other households in the asset market, but not in the goods market. For the purpose of this paper, this assumption is taken as a primitive; the markets are simply different. One way to make this distinction rigorous is by introducing dealers that intermediate asset trade; these dealers are able to verify and certify the authenticity of a bond, and they are compensated for their service with a bid-ask spread. See Appendix A.3 for details.

where $c(t)$ is the stock of consumption goods given to all consuming households and $h(t)$ is the flow of labor services each household contributes. (Since all households dislike working equally much, it is optimal for all to work the same amount.) As the marginal utility of consuming and the marginal rate of transformation from labor to consumption are always 1, the solution sets the marginal disutility of working to equal 1 at all times, too:

$$v'[h(t)] = 1 \qquad \text{and} \qquad h_t = \left(\frac{1}{\varepsilon} + \frac{1}{\alpha}\right) c_t \qquad \forall t \geq 0 \tag{1}$$

For interpretation, recall the term $\alpha n_1 = (1/\varepsilon + 1/\alpha)$ is the expected length of time to elapse between two consumption opportunities.

2.3 Household's problem

Households decide on the flow of labor effort $h(t)$, how many real balances $z(t)$ to accumulate, and how much to trade when accessing the goods or asset market. Recall that the real wage is 1, and the price of consumption in terms of real balances is also 1 (numéraire). When given a random opportunity to consume, a household with z real balances chooses to purchase $c(z) \in [0, z]$ units of consumption. When in the asset market, a household in state i with z real balances and b bonds chooses to sell $s_i(z, b) \in [-z/q, b]$ units of bonds, at the prevailing market price q. (If $s_i < 0$, then the household is a buyer of bonds.)

The household's problem is solved in detail in Appendix A.1, and I summarize the solution here.[6] Because utility is linear over consumption, the value functions will be linear as well. Specifically, let $W_i(z, b, t)$ be the value function of a household in state i with asset portfolio (z, b) at time t. This value function satisfies:

$$W_i(z, b, t) = W_i(0, 0, t) + \mu_i(t)\, z + \beta_i(t)\, b,$$

where μ and β are the costate variables associated with real balances and bonds, respectively. Given the paths of inflation ($\pi(t)$) and bond prices ($q(t)$), the costates must satisfy the following Euler equations:

$$\rho\mu_0(t) = \dot{\mu}_0(t) - \pi(t)\mu_0(t) + \varepsilon[\mu_1(t) - \mu_0(t)] + \chi \left[\frac{\beta_0(t)}{q(t)} - \mu_0(t)\right] \tag{2a}$$

$$\rho\mu_1(t) = \dot{\mu}_1(t) - \pi(t)\mu_1(t) + \alpha[1 - \mu_1(t)] \tag{2b}$$

$$\rho\beta_0(t) = \dot{\beta}_0(t) + \mu_0(t) + \varepsilon[\beta_1(t) - \beta_0(t)] \tag{2c}$$

$$\rho\beta_1(t) = \dot{\beta}_1(t) + \mu_1(t) + \alpha[\beta_0(t) - \beta_1(t)] + \chi\,[q(t)\mu_1(t) - \beta_1(t)] \tag{2d}$$

[6] The household's problem is stated in discrete time and with concave preferences, where it is easy to prove that the solution is unique. Then, I let time become continuous and utility become linear in the limit. Thus, the solution described here is to be understood as the unique limit of a nearly-linear, nearly-continuous model.

These equations have straightforward asset pricing interpretations. For example, the marginal flow value of real balances to households in state zero ($\rho\mu_0(t)$) is composed as follows: first, real balances get value from an expected capital gain ($\dot{\mu}_0(t)$); second, they lose value to inflation ($-\pi(t)\mu_0(t)$); third, they gain value in transition to state 1 ($\varepsilon[\mu_1(t) - \mu_0(t)]$); and finally, they can be used to buy bonds at price $q(t)$ if the household has access to the asset market ($\chi\left[\beta_0(t)/q(t) - \mu_0(t)\right]$). The other equations admit analogous interpretations. The term $\mu_i(t)$ in the value of bonds represents the fact that these bonds pay a flow dividend of one unit of real balances per unit of time.

If $\pi(t)$ and $q(t)$ are expected to converge to a steady state (π^s, q^s), then the only non-negative and bounded solution of the Euler equations is convergence to the steady state $(\mu_0^s, \mu_1^s, \beta_0^s, \beta_1^s)$, which is defined to be the solution of (2) with $\pi(t) \equiv \pi^s$, $q(t) \equiv q^s$, and time derivatives equal to zero.

The fact that the value functions are linear in a household's asset holdings has two important consequences. First, the choice of labor effort depends only on which state a household is in, not on their wealth or its composition. Specifically, given the values of real balances to households in state $i = 0, 1$, labor supplies satisfy:

$$v'[h_0(t)] = \mu_0(t) \quad \text{and} \quad v'[h_1(t)] = \mu_1(t) \tag{3}$$

As v is strictly convex, households with a high value of money work harder and accumulate real balances faster than those with a low value of money. And since the first-best level of labor supply satisfies $v'(h) = 1$ for everyone, the only inflation rate that achieves this benchmark is the Friedman rule: $\pi \to -\rho$ implies $(\mu_0, \mu_1) \to 1$.

Secondly, linearity implies that the spending decisions in the goods or asset market are bang-bang: depending on the price, and unless they are exactly indifferent, households either spend everything or nothing. Thus, households with a consumption opportunity will spend all their money on goods if their valuation of the goods exceeds the value of holding on to their real balances: $1 > \mu_0$. And households with an opportunity for asset trade will spend all their money to buy bonds if the household's marginal rate of substitution exceeds the market price: $\beta_i/\mu_i > q$. Conversely, households will sell all their bonds if $\beta_i/\mu_i < q$. Figure 3 illustrates an example path of a household's money holdings, as the various transitions and trading opportunities occur.

Proposition 1 verifies that trade will occur in the natural directions: households with a consumption opportunity will use it, and households in state 1 (who value money more) will sell bonds to households in state 0 (who value bonds more, *relative to real balances*). This is also the reason why Equations (2b) and (2c) do not include asset trade terms: households in state 0 do not value bonds for their ability to be sold, because they are planning to buy *more* bonds in case a trading opportunity arises.

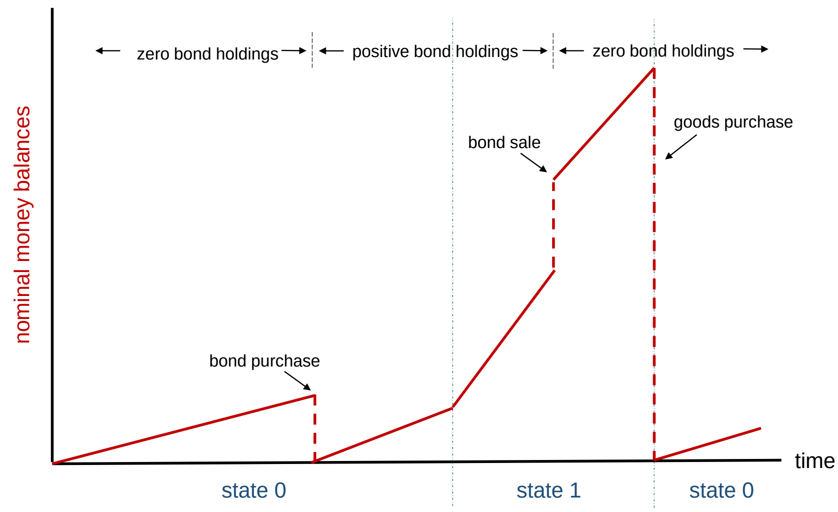


Figure 3: Sample path of a household's money holdings

Proposition 1. *In steady state, assuming $\pi > -\rho$ and $q \geq \beta_1/\mu_1$, the following inequalities hold:*

$$\mu_0 < \mu_1 < 1, \quad \frac{\beta_0}{\mu_0} > \frac{\beta_1}{\mu_1}, \quad \textit{and} \quad \frac{\beta_0}{\mu_0} > \frac{1}{\rho}.$$

In the special case of $q = \beta_1/\mu_1$, we additionally have: $\beta_1/\mu_1 < 1/\rho$.

Proof. See Appendix A.2. □

As the proposition also shows, the reservation price of bonds for households in state 0 always exceeds the "fundamental price" $1/\rho$. We can say that this reservation price exhibits a "liquidity premium" because it helps such households store their wealth for future use in a way that avoids the inflation tax. By contrast, the reservation price of bonds for households in state 1 may be above the fundamental – reflecting an anticipated future liquidity premium – or below. In the latter case, bonds carry an "illiquidity discount" for these households; they would like to liquidate their bond holdings before the consumption opportunity arrives, but with some probability will fail to do so.

2.4 Equilibrium

In equilibrium, the goods market, money market, and asset market must clear, and the government's choices must satisfy its budget constraint. In order to describe aggregate flows through the markets, let Z_i and B_i denote the *total* stocks of money and bonds held by households in state i.[7]

[7] These are totals, not averages; for example, as the overall supply of bonds is B, we have $B_0 + B_1 = B$. We would have to write $n_0 B_0 + n_1 B_1 = B$ if B_0 and B_1 were averages.

The money market clears if $Z_0 + Z_1 = \phi M$, i.e., the demand for real balances equals their supply. The goods market clears if the inflow of real balances from buyers matches the outflow to working households. As households in state 1 buy goods at flow rate α, and each such household spends all of its real balances, the flow of real balances into the market is αZ_1. In return, working households earn an income flow of h_i, so the total flow of real balances out of the goods market is $n_0 h_0 + n_1 h_1$. The equality of these flows represents the demand for goods, and thereby also determines the value of money:

$$\alpha Z_1 = n_0 h_0 + n_1 h_1 \tag{4}$$

The unconstrained flow of real balances into the bond market is χZ_0, and the unconstrained inflow of bonds is $\chi B_1 + S$ (where S represents the flow of open-market sales by the government). The ratio of the unconstrained flows defines a *candidate* market price:

$$\tilde{q} \equiv \frac{\chi Z_0}{\chi B_1 + S}$$

If buyers are willing to pay this candidate price and sellers are willing to receive it (that is, $\tilde{q} \in [\beta_1/\mu_1, \beta_0/\mu_0]$), then $\tilde{q}$ clears the bond market.[8] However, if this is not the case, then some traders must be rationed. Denote by $\psi_i \in [0, 1]$ the probability that a household in state i gets to trade; we assume for simplicity that the government never gets rationed.[9] Naturally, $\psi_i < 1$ can only be part of an equilibrium if $q = \beta_i/\mu_i$; that is, households on the long side of the market are indifferent between trading or keeping their assets. Bond market clearing can thus be expressed as equality of the constrained flows of real balances and bonds:

$$\underbrace{\chi \psi_0 Z_0}_{\text{inflow of real balances}} = \underbrace{(\chi \psi_1 B_1 + S)\, q}_{\text{outflow of real balances}}$$

with solution:

$$q = \max\left\{\frac{\beta_1}{\mu_1}, \min\left\{\tilde{q},\, \frac{\beta_0}{\mu_0}\right\}\right\}; \qquad \psi_0 = \min\left\{\frac{\beta_0}{\mu_0}\frac{1}{\tilde{q}},\, 1\right\}; \qquad \psi_1 = \min\left\{\tilde{q}\,\frac{\mu_1}{\beta_1},\, 1\right\} \tag{5}$$

The government must finance a flow of transfers T (or has access to taxes if $T < 0$) and

[8] As long as the intervention is not too large – specifically, $S \in [-\chi B_1, \chi Z_0/q]$ – it changes the magnitude of private asset flows but not their direction. Here, I assume that this is always the case, although it would in principle be possible for the government to purchase bonds at such a pace (i.e., at such an attractive price) that *everyone* would want to sell them for money. In that case, the distributional effects of the intervention would mostly go away (because everyone is a bond seller in the market, with the government on the other side). The long-term effects (of a lower bond stock) would remain the same. For the numerical experiments in Section 5, I verify numerically that the interiority assumption holds throughout.

[9] This is without loss of generality, since households will be rationed only if they are indifferent to being rationed. Indifference also means that ψ_0 and ψ_1 do not enter the household's problem, because households expect a surplus from asset trade only if they expect to get served with probability one.

dividend payments on the outstanding debt. As each unit of bonds pays a flow dividend of one unit of real balances, the total dividend flow is B. If the money supply grows at rate $\dot{M} = \gamma M$, then the government also has access to seigniorage revenue $\phi \dot{M}$, the real value of newly printed money. Using the definition of real balances, $Z \equiv \phi M$, we can express the seigniorage revenue as $\phi \dot{M} = Z \dot{M}/M = \gamma(Z_0 + Z_1)$. Finally, the government may have income from open-market sales of bonds, $q\,S$ (or must finance open-market purchases if $S < 0$). Its budget constraint is therefore:

$$T + B = \gamma(Z_0 + Z_1) + q\,S \tag{6}$$

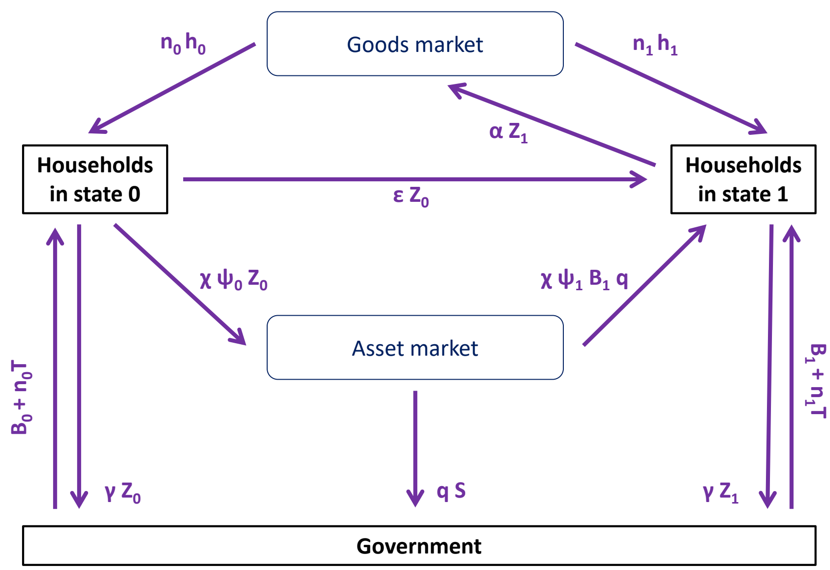


Figure 4: Flows of real balances between groups of agents

Figure 4 illustrates the flows of real balances between agents in the model, and Figure 5 does the same for bonds. Only households hold an inventory of assets, so equalizing inflows and outflows for the two markets and the government yields Equations (4), (5), and (6). What is left is to describe accumulation of assets by households. Fortunately, as explained above, all households in state $i \in \{0, 1\}$ choose identical values of labor effort, which we denote by the equilibrium per-household variables h_0 and h_1. Accounting for the flow of assets to and from households in state 0 or state 1 is then straightforward:

$$\dot{Z}_0 = -\pi Z_0 + B_0 + n_0(h_0 + T) - \varepsilon Z_0 - \chi \psi_0 Z_0 \tag{7a}$$

$$\dot{Z}_1 = -\pi Z_1 + B_1 + n_1(h_1 + T) + \varepsilon Z_0 + \chi \psi_0 Z_0 - q\,S - \alpha Z_1 \tag{7b}$$

$$\dot{B}_0 = -\varepsilon B_0 + (\alpha + \chi\psi_1)B_1 + S \tag{7c}$$

$$\dot{B}_1 = \varepsilon B_0 - (\alpha + \chi\psi_1)B_1 \tag{7d}$$

For example, the stock of real balances held by households in state 0 increases due to dividend income (B_0), labor income ($n_0 h_0$), and transfer income ($n_0 T$); it decreases via the inflation tax (πZ_0), transition to state 1 by some households (εZ_0), and expenditure on bonds in the asset market ($\chi\psi_0 Z_0$). The other equations have analogous interpretations in terms of inflows and outflows.

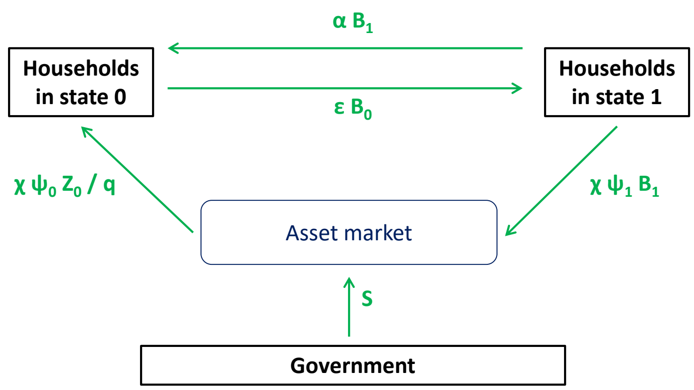


Figure 5: Flows of bonds between groups of agents

Definition 1. A strongly-monetary steady-state equilibrium is a list $\{h_0, h_1, \mu_0, \mu_1, \beta_0, \beta_1, q, \psi_0, \psi_1, Z_0, Z_1, B_0, B_1, T\}$ which satisfies Equations (3) (labor efforts are chosen optimally), (2) (the value functions represent the optimal values of bonds and money), (4) (the goods market clears), (5) (the bond market clears), $\phi = (Z_0 + Z_1)/M$ (the money market clears), (6) with $S = 0$ (the government budget is in balance), and (7) (aggregate consistency), with all time derivatives equal to zero, and in which $h_0 + T > 0$.[10]

In the definition above, government transfers T are treated as an endogenous variable that adjusts to satisfy the government budget constraint, while government debt B and the rate of money growth γ are exogenous. Since choosing γ is the purview of monetary policy,

[10] In the literature, a "monetary" equilibrium is one in which households *value* money ($\phi > 0$); here, I use the term "strongly-monetary" because the condition $h_0 + T > 0$ requires that *all households accumulate* money. (It encompasses $h_1 + T > 0$ because $\mu_1 > \mu_0$.) The reason is technical. Even if some households did not want to accumulate money, they might still value it for relaxing their budget constraint. But it can be shown that such households would *decumulate* money to hit the constraint $z \geq 0$ in finite time; as a result, their value function will not be linear in money, their willingness to pay for assets will be heterogeneous, and their decisions cannot be aggregated in the simple form shown above. To keep things simple, I focus on strong-monetaryness where value functions are linear. The condition $h_0 + T > 0$ cannot be exactly reduced to exogenous parameters, but there are some simple sufficient conditions that guarantee it: see Appendix A.2.

and since this choice imposes a constraint on the fiscal tradeoff between T and B, this assumption is one of **monetary dominance**. Alternatively, we could assume **fiscal dominance**: T and B are exogenous and γ adjusts to satisfy the budget constraint.[11]

Proposition 2. *A solution to the set of equations in Definition 1 exists and is unique. Under certain sufficient conditions on parameters, $h_0 + T > 0$ is satisfied and the solution thus describes a strongly-monetary steady-state equilibrium.*

Proof. See Appendix A.2. □

In order to define and characterize dynamic equilibria, we need to describe inflation expectations $\pi(t) = -\dot{\phi}(t)/\phi(t)$. Because $\phi \equiv Z/M$ (the real price of money equals real balances divided by the money supply), and M grows at rate γ, we can derive:

$$\pi = \gamma - \frac{\dot{Z}_0 + \dot{Z}_1}{Z_0 + Z_1}$$

We cannot just use Equations (7) to replace both $\dot{Z}_0$ and $\dot{Z}_1$, because doing so would just return the goods market clearing condition. We need to find an independent equation, and the solution is to differentiate the goods market clearing condition with respect to time:

$$\alpha \dot{Z}_1 = n_0 \dot{h}_0 + n_1 \dot{h}_1 \tag{8}$$

Next, we differentiate with respect to time Equation (3), for $i = 0, 1$:

$$\dot{h}_i = \frac{1}{v''(h_i)} \dot{\mu}_i \tag{9}$$

Finally, we can use Equations (2) to substitute for $\dot{\mu}_i$, and write:

$$\pi = \gamma - \underbrace{\left[\frac{\dot{Z}_0 + \dot{Z}_1}{Z_0 + Z_1} \right]}_{\substack{\text{all time derivatives substituted using (7a) for } \dot{Z}_0 \\ \text{and (8), (9), and (2a,b) for } \dot{Z}_1}} \tag{10}$$

With the hard work done, we can now describe a dynamic equilibrium purely in terms of ordinary differential equations, contemporaneous equations, and boundedness conditions.

[11] The literature has not yet settled on a naming convention for this important distinction. Alternative terms for monetary dominance include "active fiscal policy" and "Ricardian fiscal policy", with the counterparts for fiscal dominance being "passive fiscal policy" and "non-Ricardian fiscal policy".
Defining equilibrium under fiscal dominance is analogous; however, such an equilibrium would not be unique, because seigniorage revenue is subject to a Laffer curve. There are generally two inflation rates – a low one and a high one – that collect the same level of revenue.

Definition 2. A strongly-monetary dynamic equilibrium is a list of bounded paths $\{h_0(t), h_1(t), \mu_0(t), \mu_1(t), \beta_0(t), \beta_1(t), q(t), \psi_0(t), \psi_1(t), Z_0(t), Z_1(t), B_1(t), T(t)\}$ which satisfy Equations (2), (3), (4), (5), (6), (7), and (10), and $h_0(t) + T(t) > 0$ for all $t \geq 0$. The exogenous variables may be paths as well, provided they are bounded, piecewise continuous, and common knowledge.

3 Analysis: monetary policy with frictional asset markets

Money is clearly not superneutral: inflation affects the value of holding money, which in turn affects labor supply and production. The comparative statics of model with respect to inflation are interesting but not the main focus of the model, and are therefore relegated to Appendix A.5.

3.1 Money is not neutral in the short run

Money is neutral in the long run (since only real balances matter in equilibrium), but money injections are not neutral in the short run. A money transfer that gives everyone the same amount of currency – a "helicopter drop" – must raise prices, but thereby *compress* the distribution of real balances. The cash-rich will have a little bit less real cash, while the cash-poor will have a little bit more. Such a compression would not matter for real aggregates such as spending and production if everybody had the same valuation of money and the same propensity to spend; but it will matter whenever we are away from this knife-edge case. The key question is then how the distributions of money holdings, money valuations, and propensities to spend money interact.

In the model from Section 2, the pattern is as follows. Households in state 1 have a higher propensity to spend money on consumption goods than those in state 0 (whose propensity to spend is zero). Knowing this, households in state 1 accumulate more money than those in state zero: they work harder, and they liquidate other assets for money. These statements are true on average, but not for every individual, due to random trading histories; however, the distribution of money holdings *within* a state is irrelevant, as everyone within the same state has the same marginal valuation of money and propensity to spend it.

Translated into technical terms, in steady state it must be the case that:

$$\frac{Z_0}{n_0} < \frac{Z_1}{n_1}$$

Suppose that $N > 0$ new dollars of money are introduced lump-sum, i.e., $n_0 N$ dollars going to households in state 0 and $n_1 N$ dollars going to households in state 0. Denote the old price of money by ϕ and the new price of money by $\hat{\phi}$, and by $(\hat{Z}_0, \hat{Z}_1)$ the post-transfer levels of real balances. The previous inequality implies:

$$\frac{\hat{Z}_1}{\hat{Z}_0} = \frac{(\hat{\phi}/\phi)\, Z_1 + n_1 \hat{\phi} N}{(\hat{\phi}/\phi)\, Z_0 + n_0 \hat{\phi} N} = \frac{n_1}{n_0} \cdot \frac{Z_1/n_1 + \phi N}{Z_0/n_0 + \phi N} \quad < \quad \frac{n_1}{n_0} \cdot \frac{Z_1/n_1}{Z_0/n_0} = \frac{Z_1}{Z_0}$$

As a result, a helicopter drop of money will shift purchasing power away from state-1 households and towards state-0 households. Naturally, the increase in the money supply raises the price level $1/\phi$, but by how much? If it rises in proportion to the money increase ($\phi/\hat{\phi} = 1 + N/M$), then Z_1 falls (since state-1 households held more than the average amount of money, and their share went down), and we have a lack of demand in the goods market. If it rises only enough to keep Z_1 constant, then Z_0 exceeds its steady-state level, so we must anticipate excess inflation along the transition path. This decreases the value of money and motivates households to work less and supply fewer goods, so we have a lack of supply in the goods market. (For an illustration, see Figure 8 in Section 5 below.)

The only solution lies somewhere in the middle, depending on the elasticity of supply: the price level must undershoot its long-term path, and along the transition path the goods market clearing equation is satisfied at a lower level of consumption and production. Correspondingly, state-0 households have excess purchasing power compared to the steady state. What do they spend it on? Bonds, in the bond market. Thus, the helicopter drop also increases the bond price q along the transition path; equivalently, it lowers real bond yields.[12] This fall in yields may stimulate investment and output, indirectly, as I show in Section 4. But contrary to traditional wisdom, the helicopter drop also has a *negative direct effect* on consumption demand, output, and welfare, as real balances are less efficiently distributed.

All of these results follow from the fact that in this model, the quantity of money someone holds and their propensity to spend this money are positively correlated. This is due to two assumptions: the heterogeneous propensity to spend (α versus 0, due to the state cycle), together with the linear value function that makes the within-state distribution of money irrelevant. Most models of heterogeneous money holdings have made the opposite assumptions (Berentsen, Camera, and Waller, 2005; Molico, 2006; Chiu and Molico, 2010; Rocheteau, Weill, and Wong, 2015): all households have the same propensity to spend, but households have concave utility over money holdings, and the marginal utility of money is downward sloping. This gives rise to a risk-sharing channel: the cash-rich households value a marginal dollar less than the cash-poor ones, thus compressing the distribution of real balances – via a helicopter drop – generally results in higher spending, production, and welfare. (But not always: e.g, see Jin and Zhu (2017) where the motivation to work falls by enough to offset the increased ability to spend, and Kam and Lee (2017) where the estimated inflation tax dominates the gains from redistribution). In reality, we should expect both channels to coexist, but

[12] Alvarez and Lippi (2014) derive a very similar result – after a money injection, the price level undershoots and interest rates fall – but in an endowment economy where there is no effect on output.

an empirical analysis of which channel dominates is beyond the scope of the present paper. The fundamental question is: why do we think some people hold more money than others? Because they tried to spend it and failed? Or, because they anticipate needing it soon?

3.2 Helicopter drops versus open-market purchases

Let us next analyze a temporary intervention, where $S < 0$ for a finite period of time (an open-market purchase of illiquid bonds). For simplicity, assume that the economy was previously in a steady-state equilibrium with no expectations of intervention, and that the asset market was in the interior region where $q = Z_0/B_1$. Then, the government intervenes in the frictional bond market to buy bonds for money, driving up bond prices to $q = \chi Z_0/(\chi B_1 + S)$. During the intervention, Z_1 is higher relative to the old steady state, Z_0 and B_0 are lower, and B_1 is unaffected (to a first approximation). The rise in Z_1 implies more consumption and higher output during the intervention, but also a rise in the price level to absorb some of the extra demand. It is this rise in the price level that causes Z_0 to fall.

Once the intervention has concluded, however, and $q = Z_0/B_1$ holds again, the temporary fall in Z_0 will *depress* bond prices along the transition path to the new steady state. Because the supply of bonds available to households is lower in the new steady state than in the initial one, bond prices q will ultimately be higher. Thus there are two opposing forces on q after the intervention: the need to converge to a higher level in the new steady state, and the lack of demand for bonds due to a temporary depression of Z_0. Either one could prevail quantitatively. In summary, an open-market purchase of illiquid assets causes lower yields both during the intervention and in the long run, but possibly not along the entire transition path. In addition, the purchase has a positive direct effect on output in the short run because real balances are more concentrated in the hands of those planning to spend them, but a negative direct effect on output in the long run because scarce bonds perform a useful service in this economy.

It is instructive to contrast the effects of an open-market purchase with those of a helicopter drop analyzed above. Recall that a helicopter drop compresses the distribution of real balances, shifting purchasing power into the hands of people planning to buy goods (state-1 households) and out of the hands of people planning to buy assets (state-0 households). An open-market purchase does the opposite. It shifts purchasing power away from asset demand and towards goods demand, although total asset demand may still be increased when we add the open-market purchase itself ($S < 0$) to the remaining private demand. The direct effects on output therefore go in opposite directions, but both interventions cause lower bond yields. (For an illustration, see Figures 9-10 in Section 5 below.)

In addition to these direct effects on output, there may be indirect effects through investment: as Section 4 shows, bonds and capital are imperfect substitutes as assets, thus lower bond yields may stimulate investment and, indirectly, output.

3.3 Long-run effects of the bond supply

The comparative statics of steady-state equilibrium with respect to the bond supply B are important because they help understand the twin roles these bonds play: they are better saving vehicles than money, but they also provide indirect liquidity services because households can liquidate them when they expect to need money soon. To begin with, I assume that the money growth rate γ is fixed by monetary policy, and that the flow of lump-sum transfers, T, adjusts to satisfy the government budget constraint. A look at the asset market clearing equations (5) suggests that there are three regions to consider, and the total bond holdings by households in state 1, B_1, is a crucial variable in determining which region equilibrium falls into. In steady state, B_1 is a constant proportion of the bond supply B:

$$B_1 = \frac{\varepsilon}{\varepsilon + \alpha + \chi\psi_1} B.$$

Since $\chi\psi_1 > 0$ and since the measure of households in state 1 is $n_1 = \varepsilon/(\varepsilon + \alpha)$, households in state 1 hold fewer bonds per individual household than those in state 0.

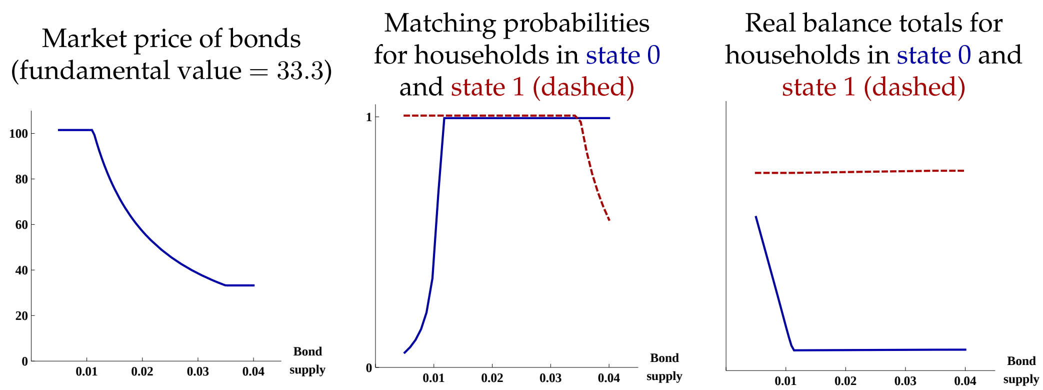


Figure 6: Comparative statics of the bond supply, under the assumption of monetary dominance and no capital in the model. Key parameters: $\rho = .03$ and $\gamma = \pi = 0$.

Figure 6 shows the effect of bond supply on bond market equilibrium. There are indeed three regions; in the first, the supply of bonds by households in state 1 is too large for the demand by households in state 0. Equilibrium is in this region if B is large, and in this case $q = \beta_1/\mu_1$ (the market price equals the reservation price of bond sellers) and $\psi_1 < 1$ (bond sellers are rationed). Within this region, changes in B have no effect on the equilibrium.

In the second region, the supply of bonds is intermediate, so that $q = Z_0/B_1$. In this case, $\psi_0 = \psi_1 = 1$ (all asset market participants are served) and $B_1 = \varepsilon/(\varepsilon + \alpha + \chi)B$, so an increase in bond supply directly decreases q. Using the Euler equations (2), we can establish that this decrease in q causes μ_0 to rise while μ_1 is unaffected; converting money into bonds becomes cheaper for households in state 0, and they are therefore willing to work harder and accumulate more money. By the goods market clearing equation (4), the extra production causes Z_1 to increase, and if the money supply has not changed, this is achieved through a fall in the price level. The end result of an increase in bond supply in this region is lower prices, higher consumption and higher output, and higher welfare.[13]

In the third region, the supply of bonds is so small that the demand by households in state 0 cannot be satisfied. Equilibrium is in this region if B is small, and in this case $q = \beta_0/\mu_0$ (the market price equals the reservation price of bond buyers) and $\psi_0 < 1$ (bond buyers are rationed). Changes in B have no effect on prices, consumption, production, or welfare, just like in the first region when the bond supply was large. In comparison to the other regions, output and welfare are lower if the bond supply is small. The intuition is that these bonds provide a useful service: they help households in state 0 store their wealth in such a way that avoids the inflation tax. As a result, such households are willing to accumulate wealth faster. Limiting the bond supply drives down yields, and may encourage households to invest in alternative assets such as physical capital (see Section 4), with an offsetting positive effect on output; but through the liquidity channel alone, a lower bond supply reduces output.

3.4 The liquidity trap, and fiscal dominance

Hence, the region of low bond supply is a *liquidity trap*. Not because the yield on bonds was constrained by some kind of technical bound, whether zero in nominal terms or otherwise; instead, the bond price is constrained by households' willingness to pay for bonds with money. Or, according to the original definition (Keynes, 1936): "almost everyone prefers cash to holding a debt which yields so low a rate of interest". The fact that the possibility of a liquidity trap arises naturally in this model, with real long-term bonds and flexible prices, illustrates what the liquidity trap does *not* require: short-term bonds, nominal bonds, a zero lower bound, or sticky prices.

In the liquidity trap region, a (tax-financed) change in the bond supply does not affect bond prices or output in the long run. However, higher expected inflation does have effects (see Appendix A.5): it would increase bond prices, and, through the liquidity channel, reduce output. (In the extended model with capital, higher inflation and higher bond prices can increase output.) Thus, an extended program of quantitative easing, whereby bonds were purchased through an expansion in the money supply, could still affect the economy even in the liquidity trap – through temporary inflation rather than through the lower stock

[13] Welfare increases because the rise in μ_0 moves equilibrium closer to the first-best. The first-best is attained if μ_0 and μ_1 are both equal to 1, the marginal utility of consumption.

of bonds.

However, all of the previous analyses assume monetary dominance: the government is committed to a certain growth rate of the money supply, and adjusts its fiscal balance to satisfy the budget constraint. While common in monetary theory, this assumption is not always realistic. Government spending is often fixed in real terms rather than dollar units: e.g., ordering ten fighter jets for whatever the final price turns out to be, rather than ordering as many fighter jets as $100 million will buy. If the government is committed to a certain fiscal balance, possibly including debt service, and money growth adjusts as required, then we are in a regime of fiscal dominance. This case is especially realistic in an environment of low inflation and interest rates; an inflation-fighting central bank can raise rates to force the fiscal authority to spend less, but it may not be able to force the fiscal authority to spend more. Then, the money growth rate is endogenously determined by the ratio of the fiscal deficit to the amount of money households are willing to hold:

$$\gamma = \frac{T + B}{Z_0 + Z_1}$$

With a representative household, there would not be much of a difference between these two regimes. But here, there is a big difference, because the *distribution* of real balances affects the *level* of real balances households end up holding in equilibrium. For example, in the liquidity trap region, the level of real balances held by households in state 0 is sensitive to changes in the supply of bonds (see the third panel of Figure 6). The reason is that households in state 0 are willing to accumulate money, but they would prefer to convert them into bonds which have a better return. Since the supply of bonds is low, they may not be able to buy bonds quickly ($\psi_0 < 1$). Therefore, the lower the aggregate bond stock, they higher is the proportion of the total money stock held by households in state 0. The next step of the argument is crucial: it is not the total money stock that determines the price level via goods market clearing, but the *money held by households looking to spend it* on goods (Equation 4). This is the big lesson from household heterogeneity. If the households about to spend money hold a lower share of it, then the velocity of circulation – and the aggregate price level – must be lower.

Consequently, a lower bond supply will increase the level of real balances households are willing to hold, and thereby reduce the inflation rate.[14] Even if the government keeps the budget deficit $T + B$ constant (it rebates the lower debt service cost to households as a transfer), so that the implied seigniorage revenue $\gamma(Z_0+Z_1)$ remains constant, the decrease in total real balances implicitly raises the inflation rate. Both of these cases are illustrated in Figure 7. As the price of bonds is increasing in inflation in the liquidity trap region (Appendix A.5), we

[14] Strictly speaking, this is only true if $T + B > 0$, i.e. the government is monetizing a deficit. The argument is reversed when the government is running a surplus and seigniorage is negative.

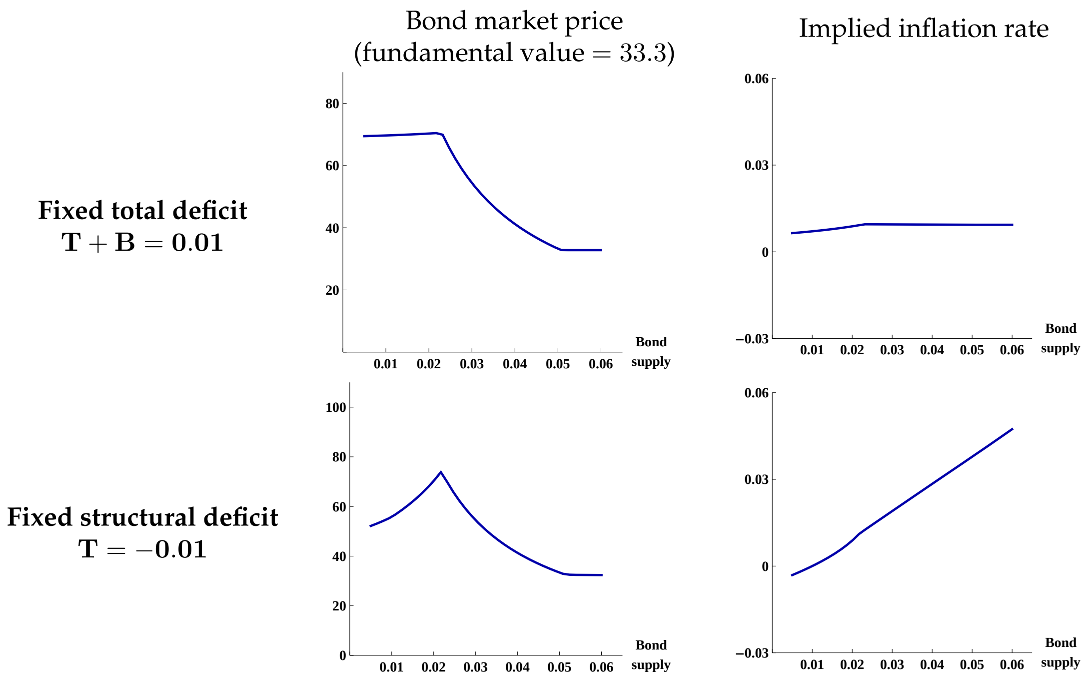


Figure 7: Comparative statics of the bond supply under fiscal dominance (inflation is endogenous). Key parameter: $\rho = .03$.

obtain the counterintuitive result that in a liquidity trap with fiscal dominance, the effective long-run demand curve for bonds is upward-sloping.

4 A model of partially liquid capital

In the previous section, we have seen several ways in which monetary policy can exert direct influence on the real economy, through temporary portfolio effects or the supply of saving vehicles. However, most economists (and their textbooks) treat these channels as secondary, and instead emphasize how monetary policy affects the economy *through the effect of interest rates on investment*. The well-known wrinkle is that yields on government bonds do not represent the cost of funds for firms looking to expand production. By introducing physical capital as a partially liquid asset the model can address this question directly.

There is now a fifth commodity: physical capital. In contrast to the other two assets (money and bonds), capital depreciates at rate $\delta > 0$; on the other hand, capital is productive. In analogy to the neoclassical growth model, there is an indeterminate measure of price-taking firms that rent labor and capital services from households on competitive spot markets, then use them to produce output according to the standard production function:

$$f(k, h) = Ak^{\xi}h^{1-\xi}, \tag{11}$$

where $\xi \in (0, 1)$ is the capital share and $A > 0$ is total factor productivity. Firms take as given the rental rate R, the real wage ω, and the output price 1 (because output is the numéraire).

Produced output can either be sold as consumption goods to the flow of households that experience the α-shock, or it can be bought by any household seeking to invest in new capital. Thus, investment is not subject to any trading friction.[15] On the other hand, investment is subject to a real friction: if $i(t)$ denotes the flow of goods invested by a household, and $k(t)$ denotes the stock of capital held by the same household, then new capital is created at a flow rate $\Phi[i(t)/k(t)]k(t)$, where Φ is a strictly concave function.

$$\dot{k}(t) = -\delta k(t) + \Phi\left(\frac{i(t)}{k(t)}\right) k(t) \tag{12}$$

The reason to include this capital accumulation friction in the model is that without it, if the value of capital were to change then households would want to jump discretely to the new equilibrium capital stock. In continuous time, that would imply infinite investment concentrated into an infinitesimal amount of time, raising technical issues. In order to sidestep these, I use the strictly concave function Φ:

$$\Phi\left(\frac{i}{k}\right) = \frac{\delta^{\nu}}{1-\nu}\left(\frac{i}{k}\right)^{1-\nu} - \frac{\delta\nu}{1-\nu}, \tag{13}$$

where δ is the rate of depreciation, and $\nu \geq 0$ is a parameter that governs how difficult it is to change the rate of capital accumulation.[16] If $\nu \to 0$, the friction disappears. This functional form has the important feature that even though the efficacy of investment is strictly concave, a household's value function still turns out to be linear in all three assets.

Just like bonds, capital cannot be used as a medium of exchange, but it can be traded in a frictional asset market. The capital market is analogous to that for bonds but perfectly segmented. In order to keep notation transparent, re-name to χ^B the arrival rate of trading opportunities in the frictional bond market, and denote by χ^K the arrival rate in the frictional capital market. Similarly, index the market outcome variables q, ψ_0, ψ_1 by superscripts B and K, respectively.

Because the real wage is no longer constant, we have to adapt Equations (3). Given the value of real balances to households in states 0 and 1, and the real wage ω, the representative choices of labor supply satisfy:

$$v'[h_0(t)] = \mu_0(t)\omega(t) \quad \text{and} \quad v'[h_1(t)] = \mu_1(t)\omega(t) \tag{14}$$

[15] See Geromichalos and Herrenbrueck (2017) for a model where consumption and investment are both subject to trading frictions.

[16] Expressing capital adjustment costs in this fashion goes back to Lucas and Prescott (1971) and is standard in the business cycle literature (e.g., Francis and Ramey, 2005).

Denote by $\kappa_i(t)$ the costate variable associated with physical capital for a household in state $i \in \{0, 1\}$. Standard Tobin's-q algebra yields that the household's rate of investment i, the value of capital, and the value of real balances are related as follows:

$$\Phi'\left(\frac{i}{k}\right) = \frac{\mu_i}{\kappa_i} \quad \text{and} \quad \rho\kappa_i = \dot{\kappa}_i + R\mu_i + \left[-\delta + \Phi\left(\frac{i}{k}\right) - \frac{i}{k}\Phi'\left(\frac{i}{k}\right)\right]\kappa_i + [\text{transitions}]$$

It follows that households invest in proportion to their capital holdings, in such a way that their capital holdings increase net of depreciation if and only if the marginal value of capital exceeds that of real balances:

$$i = \left(\frac{\kappa_i}{\mu_i}\right)^{1/\nu} \delta k. \tag{15}$$

The household-level variables i and k can be eliminated from the Euler equations for the value of capital. Consequently, value functions are linear in a household's capital holdings. Taking as given R, the rental rate on capital, and q^K, the secondary market price of capital, the value of capital to households in state 0 or 1 satisfies:

$$\rho\kappa_0 = \dot{\kappa}_0 + R\mu_0 - \frac{1 - \nu\,[\kappa_0/\mu_0]^{(1-\nu)/\nu}}{1-\nu}\,\delta\kappa_0 + \varepsilon(\kappa_1 - \kappa_0) \tag{16a}$$

$$\rho\kappa_1 = \dot{\kappa}_1 + R\mu_1 - \frac{1 - \nu\,[\kappa_1/\mu_1]^{(1-\nu)/\nu}}{1-\nu}\,\delta\kappa_1 + \alpha(\kappa_0 - \kappa_1) + \chi^K\left[q^K\mu_1 - \kappa_1\right] \tag{16b}$$

As these Euler equations show, the value of capital (and thereby, investment) is governed by two forces. First, the value of using capital in production (the rental rate R), and second, the value of capital as a liquid asset (the resale price q^K). Thus, the effect of monetary policy on investment will depend on the direction of these two forces, and if they are opposed, on which one dominates. All the results in Section 5 below can be traced to these two forces.

The Euler equation for money accumulation by households in state 0 changes, too, since these households can now use their money to buy both bonds and capital on decentralized markets. The other three of the original Euler equations do not change, but for completeness I include them here:

$$\rho\mu_0 = \dot{\mu}_0 - \pi\mu_0 + \varepsilon[\mu_1 - \mu_0] + \chi^B\left[\frac{\beta_0}{q^B} - \mu_0\right] + \chi^K\left[\frac{\kappa_0}{q^K} - \mu_0\right] \tag{17a}$$

$$\rho\mu_1 = \dot{\mu}_1 - \pi\mu_1 + \alpha[1 - \mu_1] \tag{17b}$$

$$\rho\beta_0 = \dot{\beta}_0 + \mu_0 + \varepsilon[\beta_1 - \beta_0] \tag{17c}$$

$$\rho\beta_1 = \dot{\beta}_1 + \mu_1 + \alpha[\beta_0 - \beta_1] + \chi^B\left[q^B\mu_1 - \beta_1\right] \tag{17d}$$

As before, in order to keep the value functions linear, it needs to be the case that households in state 0 do accumulate some money. Put differently, capital accumulation cannot be too rapid for any household. A simple sufficient condition would be that the real balances earned as capital income exceed the real balances invested in new capital. Since households invest in proportion to how much capital they hold, we need to make the following assumption on equilibrium outcomes:

$$R \geq (\kappa_0/\mu_0)^{1/\nu}\,\delta \tag{18}$$

Again, there is no way to write the condition purely in terms of exogenous parameters. But it will be satisfied if $\mu_0 \to \mu_1$ (in words: for low enough inflation, because then capital is not valued for its liquidity properties, and we have both $\kappa_0/\mu_0 \to 1$ and $R \to \rho + \delta$) or if $\nu \to \infty$ (households exactly invest to replace their depreciating capital, no more and no less).

4.1 Equilibrium

The aggregation of household choices follows the same principles as in Section 2. Let us begin with the aggregate flows of capital between households in state 0 and state 1. Flows due to transition between states and due to trade are standard by now, but we also have to account for accumulation. For example, a household in state 0 with capital holdings k loses a flow of δk to depreciation, but it spends i units of output on investment. This household will therefore accumulate capital at the following rate:

$$\begin{aligned}\dot{k} &= -\delta k + \Phi\left(\frac{i}{k}\right) k \;=\; -\delta k + \Phi\left(\delta(\kappa_0/\mu_0)^{1/\nu}\right) k \\ &= \frac{(\kappa_0/\mu_0)^{(1-\nu)/\nu} - 1}{1-\nu}\,\delta k\end{aligned}$$

after some algebra, and a household in state 1 will make analogous choices. Consequently, if we use K_0 and K_1 to denote the total stocks of capital held by households in states 0 and 1, respectively, then K_0 and K_1 must satisfy:

$$\dot{K}_0 = \frac{(\kappa_0/\mu_0)^{(1-\nu)/\nu} - 1}{1-\nu}\,\delta K_0 - \varepsilon K_0 + \left(\alpha + \chi^K \psi_1^K\right) K_1 \tag{19a}$$

$$\dot{K}_1 = \frac{(\kappa_1/\mu_1)^{(1-\nu)/\nu} - 1}{1-\nu}\,\delta K_1 + \varepsilon K_0 - \left(\alpha + \chi^K \psi_1^K\right) K_1 \tag{19b}$$

Next, we need to describe the accumulation of money and bonds. As in the baseline model, all households in state $i \in \{0, 1\}$ choose identical values of labor effort, which we denote by the equilibrium per-household variables h_0 and h_1. Four things change, however. First, the real wage is no longer equal to 1. Second, households now derive income from their capital holdings. Third, they also spend some of their income on investment goods; and finally, they participate in the frictional market for existing capital.

$$\dot{Z}_0 = -\pi Z_0 + B_0 + \left[R - \delta \left(\frac{\kappa_0}{\mu_0} \right)^{1/\nu} \right] K_0 + n_0(\omega h_0 + T)$$
$$\dots \qquad - \varepsilon Z_0 - \left(\chi^B \psi_0^B + \chi^K \psi_0^K \right) Z_0 \tag{20a}$$

$$\dot{Z}_1 = -\pi Z_1 + B_1 + \left[R - \delta \left(\frac{\kappa_1}{\mu_1} \right)^{1/\nu} \right] K_1 + n_1(\omega h_1 + T)$$
$$\dots \qquad + \varepsilon Z_0 + \left(\chi^B \psi_0^B + \chi^K \psi_0^K \right) Z_0 - q^B S - \alpha Z_1 \tag{20b}$$

$$\dot{B}_0 = -\varepsilon B_0 + \left(\alpha + \chi^B \psi_1^B \right) B_1 + S \tag{20c}$$

$$\dot{B}_1 = \varepsilon B_0 - \left(\alpha + \chi^B \psi_1^B \right) B_1 \tag{20d}$$

The market clearing conditions for the bond market are the same as before, given by Equations (5), but now with ψ_0, ψ_1, and q indexed by superscript B. The market clearing conditions for the frictional capital market are analogous. The ratio of asset inflows ($\chi^K Z_0$ of real balances and $\chi^K K_1$ of capital) defines a candidate price $\tilde{q}^K$; the actual price will equal the candidate price unless it exceeds the price bounds that households are willing to pay/receive. And in the latter case, the long side of the market will be rationed with probability ψ_i^K:

$$q^K = \max\left\{ \frac{\kappa_1}{\mu_1}, \min\left\{ \tilde{q}^K, \frac{\kappa_0}{\mu_0} \right\} \right\}; \quad \psi_0^K = \min\left\{ \frac{\kappa_0}{\mu_0} \frac{1}{\tilde{q}^K}, 1 \right\}; \quad \psi_1^K = \min\left\{ \tilde{q}^K \frac{\mu_1}{\kappa_1}, 1 \right\} \tag{21}$$

Goods market clearing now includes accounts for investment:

$$\alpha Z_1 + \left(\frac{\kappa_0}{\mu_0} \right)^{1/\nu} \delta K_0 + \left(\frac{\kappa_1}{\mu_1} \right)^{1/\nu} \delta K_1 = A \left(K_0 + K_1 \right)^{\xi} \left(n_0 h_0 + n_1 h_1 \right)^{1-\xi} \tag{22}$$

The spot markets for capital and labor services clear at competitive prices:[17]

$$R = A\xi \left(\frac{K_0 + K_1}{n_0 h_0 + n_1 h_1} \right)^{\xi - 1} \qquad\qquad \omega = A(1-\xi) \left(\frac{K_0 + K_1}{n_0 h_0 + n_1 h_1} \right)^{\xi}$$

The system of capital flows (19) is linear homogeneous in (K_0, K_1). Consequently, $K_0 = K_1 = 0$ is one steady-state solution to this system, but it would imply that there is no production, and hence no monetary economy. But because of homogeneity, all steady-state solutions satisfy that the determinant of the system is zero, and from this we can derive two steady-state equations:

[17] As an alternative, one could model a frictional labor market where firm entry determines employment (as done by Berentsen, Menzio, and Wright, 2011; Rocheteau and Rodriguez-Lopez, 2014; Dong and Xiao, 2017).

$$\left[\left(\frac{\kappa_0}{\mu_0}\right)^{(1-\nu)/\nu}-1\right](\alpha+\chi^K\psi_1^K)=\left[1-\left(\frac{\kappa_1}{\mu_1}\right)^{(1-\nu)/\nu}\right]\left[\varepsilon-\delta\frac{(\kappa_0/\mu_0)^{(1-\nu)/\nu}-1}{1-\nu}\right] \tag{23a}$$

$$\left[\left(\frac{\kappa_0}{\mu_0}\right)^{(1-\nu)/\nu}-1\right]K_0=\left[1-\left(\frac{\kappa_1}{\mu_1}\right)^{(1-\nu)/\nu}\right]K_1 \tag{23b}$$

The first equation makes the determinant of the right-hand side of the system (19) zero. The second equation defines the kernel of the system, i.e. the combinations of capital stocks held by households in states 0 and 1 that are possible solutions. Clearly, the ratios κ_0/μ_0 and κ_1/μ_1 are key. They represent Tobin's Q, the marginal rate of substitution between using real income for investment or other purposes, for households in states 0 and 1. Equation (23b) implies that the ratios κ_0/μ_0 and κ_1/μ_1 must be on opposite sides of 1 in steady state. And according to Equation (15), households invest enough to make up for depreciation only if their ratio κ/μ exceeds 1. We have already learned from the baseline model that households in state 1 value money more relative to any other asset than households in state 0. Thus, households in state 1 will invest less than necessary to make up for depreciation, while households in state 0 will invest more, so that the aggregate stock of capital remains constant.

If $\nu\to 0$, capital accumulation becomes frictionless. As we have seen, this is no problem for the existence and computation of steady states but it greatly complicates dynamics. Nevertheless, it provides a benchmark. It would still be the case that $\kappa_0/\mu_0>\kappa_1/\mu_1$, i.e. households in state 0 would value capital more than households in state 1. Without the accumulation friction, Tobin's Q must equal 1 for every agent choosing to accumulate capital. Accordingly, we must have $\kappa_0/\mu_0=1>\kappa_1/\mu_1$, and only households in state 0 accumulate capital.

Definition 3. A strongly-monetary steady-state equilibrium with capital is a vector $\{h_0,h_1,\mu_0,\mu_1,\beta_0,\beta_1,\kappa_0,\kappa_1,q^B,\psi_0^B,\psi_1^B,q^K,\psi_0^K,\psi_1^K,Z_0,Z_1,B_0,B_1,K_0,K_1,R,\omega,T\}$ which satisfies Equations (17), (16), (14), (5), (6), and (20)-(23) with time derivatives equal to zero and $S=0$, and in which the two conditions (18) and $\omega h_0+T>0$ hold.

The fully dynamic equilibrium is more complex, as was the case for the version of the model without physical capital: the price level in the extended model is determined by Equation (22), and expected inflation will not be equal to money growth along the transition path.

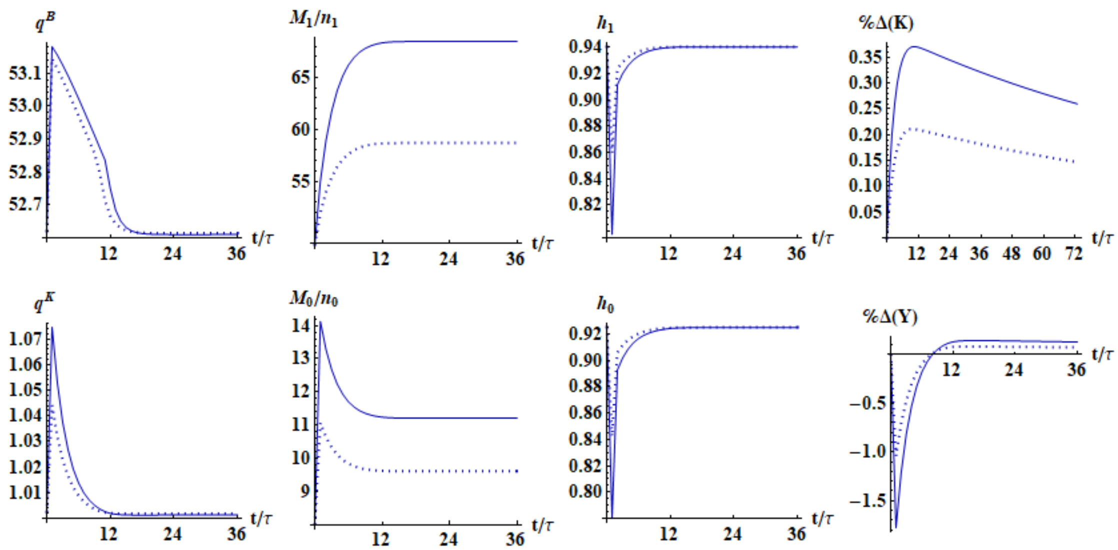
The solution is to redo the construction from Equations (8) to (10). The latter equation stays unchanged:

$$\pi=\gamma-\frac{\dot Z_0+\dot Z_1}{Z_0+Z_1}$$

We need to differentiate Equation (22) with respect to time, substitute $\dot h_i$ using Equation (9), substitute the remaining time derivatives using the dynamic equations for the costate and aggregate state variables, and then use the result to substitute for $\dot{Z}_1$. We finally substitute $\dot{Z}_0$ using Equation (20a), and we are left with an equation describing expected inflation purely as a combination of contemporaneous variables. The definition of dynamic equilibrium is the obvious extension of Definitions 2 and 3.

5 Analysis: monetary policy and investment

We continue the analysis of monetary interventions from Section 3, and apply its lessons to the model with capital. Again, begin with considering a helicopter drop, shown in Figure 8.



Continuous line: 40% increase in the money stock. Dotted line: 20% increase

Figure 8: Dynamics after a one-time helicopter drop. The drop is unanticipated in period 0 and happens in period 1. Showing: the price of bonds (fundamental: $1/\rho = 33.3$), the price of capital (fundamental: 1), the money held per household in state 0 and 1, the labor supply by households in state 0 and 1 (first-best: 1), all in levels. Finally, the capital stock and output are shown in percent deviations from the initial level.

The money injection increases everybody's money holdings, but proportionally more so for households in state 0 (second column of the figure). Thus, demand for illiquid assets is stimulated, increasing their prices in the secondary markets (first column). The flip side is that demand for consumption goods is depressed, which results in lower labor supply (third column) and output (fourth column, bottom row); this is the short-term component of the "liquidity channel" analyzed in Section 3. However, now there is an additional, medium-term, "investment channel" present: the elevated price of capital causes an investment boom (K/Y, fourth column, top row). As capital is a slow moving variable, the increase in the capital stock persists for much longer than the portfolio effects do, which results in elevated output in the medium run before ultimately returning to the original steady state.

Since bonds are real, a change in the money supply is neutral in steady state. This would be different if bonds were nominal, as then the increase in the money-bonds ratio would lower the *real* supply of bonds.

The effects of a reduction in the real bond supply can be seen in Figures 9-10, which illustrate the dynamic equilibrium following a sustained program of bond purchases.[18] As the figures show, the results strongly depend on parameters which govern whether a particular channel dominates. The parameters for the two figures were chosen as follows:

Figure 9: Capital is scarce in the secondary market ($q^K = \kappa_0/\mu_0$, the upper bound), which makes investment more sensitive to bond prices. Labor supply is inelastic, which weakens the liquidity channel.

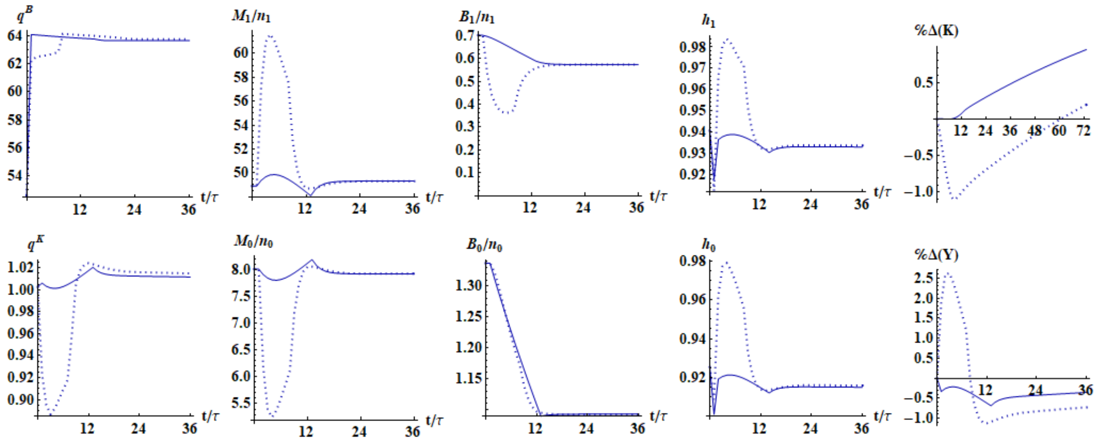
Figure 10: Capital is plentiful in the secondary market ($q^K = \kappa_1/\mu_1$, the lower bound), which makes investment less sensitive to bond prices. Labor supply is elastic, which strengthens the liquidity channel.

Some results are common between the two examples. In the short run, it is always the case that the bond purchases increase the prices of bonds, direct money into the hands of state-1 households, and direct money out of the hands of state-0 households. Labor falls on impact but then experiences a temporary boom, due to the extra demand for consumption goods.

Other results differ sharply. When bonds and capital are plentiful (Figure 10), the capital price falls during the bond purchases (because state-0 households, who would be buying the capital, have less purchasing power) – causing an investment "hangover" – but is unaffected in the long run (because the lower stock of bonds is not reflected in a higher bond price in the plentiful region). When capital is scarce, on the other hand, the bond purchases lead to a higher price of capital and a higher capital stock in the long run, which may or may not be enough to overcome the temporary hangover (bottom left and top right of Figure 9).

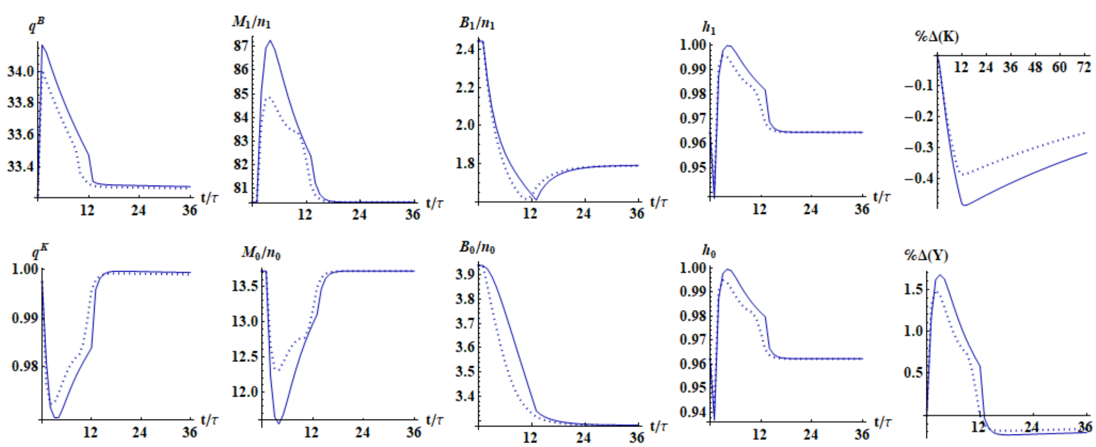
A final comparison we can make is between uniform and tapered purchases. As Figure 10 shows, there is little difference between the two as long as bonds are plentiful in the long run. As the reallocation of money balances is less drastic under the tapered program, the effects on investment and output are slightly weaker. However, this is not the case when bonds and capital are on the scarce side (Figure 9). In that regime, uniform purchases reallocate money very little, whereas tapered purchases have a strong effect. As a result, the

[18] Specifically, the experiment is as follows. In period 0, the economy is in steady state. In period 1, a bond purchase program is announced, whereby the government will buy back 20% of the outstanding bond supply in the secondary market. The purchases will be tax-financed, so that the long-run money supply is constant. While the announcement is a complete surprise, the resulting equilibrium and transition dynamics are immediately understood by everyone. I consider two variations of the purchases: a "uniform" program, where the flow of purchases is constant over 12 periods (i.e., purchasing $20\%/12 \approx 1.66\%$ of outstanding bonds every period), and a "tapered" program, where the flow of purchases starts fast (i.e., purchasing $2 \times 20\%/12 \approx 3.33\%$ of bonds in period 1) and declines to zero linearly (i.e. purchasing 0.3% of bonds in period 11 and nothing thereafter).



Continuous line: 20% decrease in the bond stock, purchased uniformly over 12 periods
Dotted line: "tapered" purchase, starting at twice the pace and declining to 0 linearly

Figure 9: Dynamics after a 12-period bond purchase program, financed with lump-sum taxes, under parameters where capital is scarce. The program is unanticipated in period 0 and fully understood from period 1 on. Showing: the bond price (fundamental: $1/\rho = 33.3$), the price of capital (fundamental: 1), the money and bonds held per household in state 0 and 1, the labor supply by households in state 0 and 1 (first-best: 1), all in levels. Finally, the capital stock and output are shown in percent deviations from the initial level.



Continuous line: 20% decrease in the bond stock, purchased uniformly over 12 periods
Dotted line: "tapered" purchase, starting at twice the pace and declining to 0 linearly

Figure 10: Dynamics after a 12-period bond purchase program, financed with lump-sum taxes, under parameters where bonds and capital are plentiful. The program is unanticipated in period 0 and fully understood from period 1 on. Showing: the price of bonds (fundamental: $1/\rho = 33.3$), the price of capital (fundamental: 1), the money and bonds held per household in state 0 and 1, the labor supply by households in state 0 and 1 (first-best: 1), all in levels. Finally, the capital stock and output are shown in percent deviations from the initial level.

outcomes related to the short-run liquidity channel – output boom combined with investment hangover – are much more pronounced than those related to the investment channel, which only assert their eventual dominance (higher capital prices and capital stock) over the long run (in the example, 60 periods after the intervention).

6 Implications for quantitative easing policies

What does the theory developed in this paper say about the possible effects of quantitative easing? To summarize the results of the previous sections: temporary open-market purchases of long-term government bonds tend to reduce the yields on these bonds and, indirectly, on other assets such as physical capital, and can thereby stimulate capital accumulation and output. Whether this effect has quantitative power will depend on a number of factors, such as the degree of asset market integration, the elasticity of investment with respect to the price of capital, and the wage elasticity of the labor supply. But on the whole, the conclusion is that quantitative easing can work, and in fact do so through the same channels as 'standard' monetary policy. However, the model also suggests three ways in which a quantitative easing program may fail or even backfire.

First, one of the ways the program works is because purchases of illiquid assets direct money into the hands of households likely to spend it on new goods and services, and out of the hands of households seeking to save, i.e. spend money on assets. When the program ends, private demand for financial assets will therefore be depressed below its long-run equilibrium level, crowded out by excess demand for assets by the central bank, leading to a "hangover" of higher interest rates and slower investment after the stimulus is withdrawn.

Second, real assets have a positive rate of return and therefore help households store their wealth more efficiently than in the form of money. Removing such assets from circulation also removes this benefit, and households who save – because they do not expect to need liquid money soon – are particularly sensitive to this effect. Reducing the supply of illiquid bonds has therefore a long-term economic cost which has to be balanced against any gains from increased capital accumulation.[19] In fact, when I calibrate the model to US data (see the Web Appendix), I find that the negative liquidity effect is likely to outweigh the increase in capital intensity, leading to *lower* output in the long run.[20]

Third, the intervention will also affect the flow of assets between households. These asset flows matter in ways that a representative household model cannot capture. For example, the model features a liquidity trap regime: due to fundamentals (preferences and market

[19] Williamson (2012) identified the lower long-run supply of liquid assets as the main cost of quantitative easing policies, but did not allow for possibly offsetting gains from capital accumulation.

[20] In reality, the QE programs in Japan and the US – but not in Europe – coincided with fiscal expansions that increased the stock of long-term debt. So a secondary implication of the calibration is that these fiscal expansions had long-term *benefits* for the economy, albeit for monetary reasons rather than fiscal ones.

structure), asset prices are inelastic to asset supply at an elevated level. In this regime, open market operations can be ineffective or even counterproductive: households will hold on to the newly created money instead of spending it, which reduces the velocity of circulation and, ultimately, the price level. This will cause at least temporary disinflation, and furthermore the lower velocity of circulation soaks up future money growth and may thereby *increase* real interest rates, reduce capital accumulation, and contract the economy.[21] This result is especially relevant to the current policy discussion because it resembles the original conception of a "liquidity trap" as a region where the relative demand of bonds and money is flat (Keynes, 1936; Robertson, 1940); having been derived in a model where bonds are real and prices are perfectly flexible and determined in competitive spot markets, it strongly suggests that the existence of a liquidity trap has little to do with price stickiness or the zero lower bound on nominal interest rates.

7 Conclusion

The model contributes to the theory of money and asset markets in important ways. For one, it is parsimonious: the only frictions are trading delays in asset markets and the fact that money is occasionally necessary to purchase consumption goods. Households are heterogeneous and their individual portfolios depend on history, but this is a feature rather than a friction. Financial assets are real and long-term, thereby abstracting from yield curve or inflation risk effects.[22] Even in this simple framework, money is not neutral in the short run, government intervention in asset markets has persistent effects, and the supply of illiquid assets matters for the macroeconomy.

A second advantage of the model is that by taking asset markets seriously, it offers new insights into the effect of monetary policy on the economy. In contrast to the overwhelming majority of the literature, open-market operations can be modeled realistically as intervention in asset markets (rather than as directly manipulating households' budget constraints), and the difference matters. The liquidity channel and the investment channel can be cleanly distinguished theoretically and empirically (although future empirical work will be able to refine the estimation); while the literature on each channel in isolation is vast, as of this point the only papers to incorporate both are Rocheteau and Rodriguez-Lopez (2014), Geromichalos and Herrenbrueck (2017), and this paper.

[21] Most monetary models assume that seigniorage revenue is kept proportional to the money stock, generating a constant and exogenous rate of money growth – monetary dominance. The alternative assumption of fiscal dominance – seigniorage revenue is fixed in real terms – is equally realistic, especially in a low inflation regime.

[22] Finite-term bonds can be liquidated in two ways: by selling them in the market, or by letting them mature. Since short-term bonds mature sooner than long-term bonds, they have inherently different liquidity properties. Geromichalos, Herrenbrueck, and Salyer (2016) analyze the implications of this fact for bond prices and the yield curve, and Williamson (2016) studies open market operations designed to twist the yield curve.

The long-term effects of permanent open-market purchases are similar to those of higher long-run inflation, and they are a combination of two opposing forces: the liquidity channel and the investment channel. If labor supply is inelastic with respect to the marginal value of income, and capital is scarce enough to be valued for its liquidity properties at the margin, then these policies can stimulate capital accumulation because capital is an alternative to money as a store of value. However, if the labor supply is elastic and capital is abundant, then these policies reduce both the capital stock and output.

An important qualification of the results is that as in most monetary-search models, higher output and capital accumulation do not necessarily reflect higher welfare. In fact, the Friedman rule is the first-best policy here. However, the list of realistic model ingredients that could alter this result is extensive: if lump-sum taxes are not available (Hu, Kennan, and Wallace, 2009; Andolfatto, 2013), if the government has an advantage in providing certain goods and services and needs taxes to finance them, if a fraction of money is spent on socially worthless activity (Williamson, 2012), or if some agents have market power, then the welfare properties of the model could be very different without affecting its ability to explain how government interventions in asset markets affect asset prices and the broader economy.

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Appendix

A.1 Discrete-time formulation of the household's problem

A.1.1 Environment (without capital)

Time $t = \{0, \tau, 2\tau, \ldots\}$ is discrete and proceeds in periods of length τ. Each period consists of two subperiods, named AM and PM ("asset market" and "production market"). The consumption good c can only be consumed at certain random opportunities revealed to households at the beginning of a PM subperiod. Labor effort is expended by every household in the PM, at a rate of τh per period. Both of these commodities are perishable and generate utility. The other two commodities are assets: they are perfectly durable and do not generate utility. Bonds pay a flow dividend of τ units of real balances per period.

At the beginning of a period, in the AM, asset market participation is revealed: a measure

$\tilde{\chi} \equiv [1 - \exp(-\chi\tau)]$ of households is randomly selected to participate in the bond market. The complementary measure $\exp(-\chi\tau)$ of households is idle and goes directly into the PM.

At the beginning of the PM subperiod, consumption opportunities and state transitions are revealed: a measure $\tilde{\alpha} \equiv [1 - \exp(-\alpha\tau)]$ of households in state 1 is selected to be consumers and to transition into state 0 in the next period. For simplicity, assume that current consumers can only spend the real balances they held in the *beginning* of the PM, but cannot use any labor income they earn in the same period until next period (when they know they will start anew from state 0). Correspondingly, a measure $\tilde{\varepsilon} \equiv [1 - \exp(-\varepsilon\tau)]$ of households in state 0 is selected to transition into state 1 in the next period. Transfers from the government are delivered (or, taxes are assessed) at the end of a period after goods trade has concluded. As in the main text, we assume that the measures of households in state 0 and 1 are constant at their steady-state levels: $n_0 = 1 - n_1 = \tilde{\alpha}/(\tilde{\varepsilon} + \tilde{\alpha})$.

Households have the following utility:

$$U(c, h) = \sum_{t=0}^{\infty} e^{-\rho t} \Big[-\tau v(h_t) + \mathbb{I}\{\text{consuming}\} u(c_t) \Big]$$

where h_t is denominated at an annual rate, so that τh_t is the labor expended per period, but consumption is denominated at a per-period rate (since in the continuous time limit, consumption opportunities will be rare). The function $v(h)$ is strictly increasing and concave and satisfies $v(0) = v'(0) = 0$ and $v'(\bar{h}) > 1$. Utility over consumption is the special function:

$$u(c) = \frac{1}{\sigma}[1 - \exp(-\sigma c)],$$

which is strictly increasing, strictly concave, and bounded, and satisfies $u'(0) = 1$.

At the end of the analysis, we will take the limits $\tau \to 0$ (continuous time) and $\sigma \to 0$ (linear utility and value functions). (Starting with a concave problem makes proving uniqueness of the solution straightforward.)

A.1.2 Household's problem

Begin with the PM subperiod, after the transition shocks $\tilde{\varepsilon}$ and $\tilde{\alpha}$ have been revealed. Since current-period income cannot be used to pay for current-period consumption, what matters for the labor choice is whether a household *will be* in state 0 or state 1 in the following period. Thus, we define end-of period value functions, W_0 and W_1, for households *exiting* a period and being in state $i = 0, 1$ in the *next* period. Similarly, define the end-of-subperiod values for households exiting the AM, before they enter the PM, by $\hat{W}_0$, $\hat{W}_1$.

At the beginning of the AM subperiod, some households are randomly selected to be trading in the bond market. Denote their (beginning-of-subperiod) values by Ω_0 and Ω_1, respectively. All these value functions must satisfy the following Bellman equations:

$$W_0(z_t, b_t) = e^{-\rho\tau}(1-\tilde{\chi})\hat{W}_0\left(\frac{\phi_{t+\tau}}{\phi_t}z_t, b_t\right) + e^{-\rho\tau}\tilde{\chi}\Omega_0\left(\frac{\phi_{t+\tau}}{\phi_t}z_t, b_t\right) \tag{A.1a}$$

$$W_1(z_t, b_t) = e^{-\rho\tau}(1-\tilde{\chi})\hat{W}_1\left(\frac{\phi_{t+\tau}}{\phi_t}z_t, b_t\right) + e^{-\rho\tau}\tilde{\chi}\Omega_1\left(\frac{\phi_{t+\tau}}{\phi_t}z_t, b_t\right) \tag{A.1b}$$

$$\hat{W}_0(z_t, b_t) = (1-\tilde{\varepsilon})\max_{h_t\in[0,\bar{h}]}\left\{-\tau v(h_t) + W_0[z_t + \tau(h_t + b_t + T_t), b_t]\right\} \tag{A.1c}$$

$$+\,\tilde{\varepsilon}\max_{h_t\in[0,\bar{h}]}\left\{-\tau v(h_t) + W_1[z_t + \tau(h_t + b_t + T_t), b_t]\right\} \tag{A.1d}$$

$$\hat{W}_1(z_t, b_t) = (1-\tilde{\alpha})\max_{h_t\in[0,\bar{h}]}\left\{-\tau v(h_t) + W_1[z_t + \tau(h_t + b_t + T_t), b_t]\right\} \tag{A.1e}$$

$$+\,\tilde{\alpha}\max_{\substack{h_t\in[0,\bar{h}]\\ c_t\in[0,z_t]}}\left\{u(c_t) - \tau v(h_t) + W_0[z_t - c_t + \tau(h_t + b_t + T_t), b_t]\right\} \tag{A.1f}$$

$$\Omega_0(z_t, b_t) = \max_{s_t\in[-z_t/q_t, b_t]}\left\{\hat{W}_0(z_t + q_t s_t, b_t - s_t)\right\} \tag{A.1g}$$

$$\Omega_1(z_t, b_t) = \max_{s_t\in[-z_t/q_t, b_t]}\left\{\hat{W}_1(z_t + q_t s_t, b_t - s_t)\right\} \tag{A.1h}$$

There is no aggregate risk, and the idiosyncratic risk of shocks has been written out explicitly. Note that h_t is written as an annual rate (labor supply per period is τh_t), but c_t is written in per-period terms (because as $\tau \to 0$, consumption will become a rare event). Note also that a household's trading action in the bond market is written as a sale, s_t. If a household buys bonds, s_t will be negative. The constraints on bond sales, bond purchases, and consumption reflect the assumption that households cannot create money or bonds, and that only money is the medium of exchange in the goods market. The split maximization problems in the PM reflect the fact that households, at the point of trading, already know whether they will be in state 0 or 1 in the next period, and can make their labor decision conditional on the future state rather than the present one.

Proposition A.1. *The household's problem has a unique solution, consisting of functions* $\{W_0, W_1, \hat{W}_0, \hat{W}_1, \Omega_0, \Omega_1\}$ *which solve Equations* (A.1). *These functions are all strictly increasing, strictly concave, and continuously differentiable.*

Proof. This is a convex problem with a bounded, differentiable, and strictly concave return function (when written in terms of consumption and leisure, $(c, \bar{h} - h)$. Hence, the theorems of Stokey and Lucas (1989), Chapter 4.2, apply. □

Taking first-order conditions, and using $h_{i,t}$ to denote the labor effort of a household that *anticipates being* in state $i = 0, 1$ in the subsequent period, we obtain:

$$v'(h_{0,t}) = \partial_z W_{0,t} \quad \text{and} \quad v'(h_{1,t}) = \partial_z W_{1,t}$$

The conditions on v' assure that this solution will always be interior. By contrast, consumption solves:

$$\partial_z W_{0,t} = \exp(-\sigma c_t)$$

in the interior, but we will have $c_t = z_t$ if $\partial_z W_{0,t} < \exp(-\sigma z_t)$. This will be the case after we take the limit $\sigma \to 0$, since it will turn out that $\partial_z W_{0,t} < 1$ (Proposition 1).

Finally, the bond offer by a household in state i solves:

$$\partial_b \Omega_{i,t} = q_t \, \partial_z \Omega_{i,t}$$

unless the household is at a corner:

$$s_{i,t} = \begin{cases} -z_t/q_t & \text{if } q_t < \partial_b \Omega_{i,t}/\partial_z \Omega_{i,t} \\ b_t & \text{if } q_t > \partial_b \Omega_{i,t}/\partial_z \Omega_{i,t} \end{cases}$$

A household cannot offer to sell more than all of her bonds, b_t. A household that offers a negative quantity of bonds for sale is of course asking to buy bonds with money. In this case, she cannot spend more than all of her real balances, z_t.

A.1.3 Deriving the Euler equations

In order to shorten notation, define the costate variables $(\mu_{i,t}, \beta_{i,t})$ to be the marginal values of real balances and bonds at the end of a period t, for a household who will be in state $i = 0, 1$ in the *following* period $t + \tau$:

$$\mu_{i,t} \equiv \partial_z W_{i,t} \quad \text{and} \quad \beta_{i,t} \equiv \partial_b W_{i,t}$$

Taking envelope conditions of the end-of-period values W_i:

$$\mu_{0,t} = e^{-\rho\tau} \frac{\phi_{t+\tau}}{\phi_t} \left[(1-\tilde{\chi}) \partial_z \hat{W}_{0,t+\tau} + \tilde{\chi} \partial_z \Omega_{0,t+\tau} \right] \quad \beta_{0,t} = e^{-\rho\tau} \frac{\phi_{t+\tau}}{\phi_t} \left[(1-\tilde{\chi}) \partial_b \hat{W}_{0,t+\tau} + \tilde{\chi} \partial_b \Omega_{0,t+\tau} \right]$$
$$\mu_{1,t} = e^{-\rho\tau} \frac{\phi_{t+\tau}}{\phi_t} \left[(1-\tilde{\chi}) \partial_z \hat{W}_{1,t+\tau} + \tilde{\chi} \partial_z \Omega_{1,t+\tau} \right] \quad \beta_{1,t} = e^{-\rho\tau} \frac{\phi_{t+\tau}}{\phi_t} \left[(1-\tilde{\chi}) \partial_b \hat{W}_{1,t+\tau} + \tilde{\chi} \partial_b \Omega_{1,t+\tau} \right]$$

Taking envelope conditions of the AM values Ω_i and substituting the AM trade solution:

$$\partial_z \Omega_{0,t} = \frac{1}{q_t} \partial_b \hat{W}_{0,t} \qquad \partial_b \Omega_{0,t} = \partial_b \hat{W}_{0,t}$$
$$\partial_z \Omega_{1,t} = \partial_z \hat{W}_{1,t} \qquad \partial_b \Omega_{1,t} = q_t \, \partial_z \hat{W}_{0,t}$$

Taking envelope conditions of the middle-of-period values $\hat{W}_i$ and – in case of a consuming household – substituting the optimal consumption condition:

$$\partial_z \hat{W}_{0,t} = (1-\tilde{\varepsilon})\mu_{0,t} + \tilde{\varepsilon}\mu_{1,t} \qquad \partial_b \hat{W}_{0,t} = (1-\tilde{\varepsilon})\beta_{0,t} + \tilde{\varepsilon}\beta_{1,t}$$
$$\partial_z \hat{W}_{1,t} = (1-\tilde{\alpha})\mu_{1,t} + \tilde{\alpha}u'(c_t) \qquad \partial_b \hat{W}_{1,t} = (1-\tilde{\varepsilon})\beta_{1,t} + \tilde{\varepsilon}\beta_{0,t}$$

And, finally, putting everything together in the Euler equations for money and bonds:

$$e^{\rho\tau}\frac{\phi_{t-\tau}}{\phi_t}\cdot\mu_{0,t-\tau} = (1-\tilde{\chi})\left[(1-\tilde{\varepsilon})\mu_{0,t} + \tilde{\varepsilon}\mu_{1,t}\right] + \tilde{\chi}\,\frac{1}{q_t}\,\left[(1-\tilde{\varepsilon})\beta_{0,t} + \tilde{\varepsilon}\beta_{1,t}\right] \tag{A.2a}$$
$$e^{\rho\tau}\frac{\phi_{t-\tau}}{\phi_t}\cdot\mu_{1,t-\tau} = (1-\tilde{\alpha})\mu_{1,t} + \tilde{\alpha}u'(c_t) \tag{A.2b}$$
$$e^{\rho\tau}\frac{\phi_{t-\tau}}{\phi_t}\cdot\beta_{0,t-\tau} = (1-\tilde{\varepsilon})\beta_{0,t} + \tilde{\varepsilon}\beta_{1,t} \tag{A.2c}$$
$$e^{\rho\tau}\frac{\phi_{t-\tau}}{\phi_t}\cdot\beta_{1,t-\tau} = (1-\tilde{\chi})\left[(1-\tilde{\varepsilon})\beta_{1,t} + \tilde{\varepsilon}\beta_{0,t}\right] + \tilde{\chi}\,q_t\,\left[(1-\tilde{\alpha})\mu_{1,t} + \tilde{\alpha}u'(c_t)\right] \tag{A.2d}$$

Together with the transversality conditions:

$$\lim_{t\to\infty} e^{-\rho t}\mu_{0,t}z_t = 0 \qquad \lim_{t\to\infty} e^{-\rho t}\beta_{0,t}b_t = 0$$
$$\lim_{t\to\infty} e^{-\rho t}\mu_{1,t}z_t = 0 \qquad \lim_{t\to\infty} e^{-\rho t}\beta_{1,t}b_t = 0,$$

these Euler equations characterize every optimal path of asset accumulation.

A.1.4 Linearity in the limit as $\sigma \to 0$

Any individual household must have a finite amount of wealth (z, b) (although it can get arbitrarily large due to idiosyncratic shocks). Since consumption c is bounded by z, the desired level of consumption must be finite for every household, too. Thus, we can take the limit as $\sigma \to 0$:

$$u'(c) = \exp(-\sigma c) \to 1$$

Evaluating the Euler equations in steady state reveals that as long as $\exp(\rho\tau)\phi_{t-\tau} < \phi_t$ (the value of money does not grow faster than at rate ρ), we must have $\mu_0 < \mu_1 < 1$. As a consequence, in the limit $\sigma \to 0$ every household spends all of their money, and the Euler equations become independent of the household's portfolio. As long as the solution to the labor supply equations implies that households *accumulate* money – so that a particular household's distance to holding zero money is irrelevant – the optimal path solution of the Euler equations is also independent of a household's portfolio. Since the costate variables are the partial derivatives of the value functions, the value functions must be linear.

This also implies that the asset trading solution must be bang-bang: households spend either everything or nothing, and as long as $\beta_1/\mu_1 < \beta_0/\mu_0$ (see the proof in the next subsection), this means that households in state zero buy bonds with money, and households in state 1 sell bonds for money.

A.1.5 The continuous-time limit as $\tau \to 0$

With some algebra, and using the facts that $\tilde{\chi} \to 0$ but $\tilde{\chi}/\tau \to \chi$ as $\tau \to 0$ (and so on for the other shock probabilities), the Euler equations converge to exactly the result from the main text (Equations 2).

A.2 Proofs of statements

Proof. Proof of Proposition 1. Impose steady states, and write the equations in general form (without knowing the direction of asset trade):

$$
\begin{aligned}
(\rho+\gamma)\mu_0 &= \varepsilon(\mu_1-\mu_0) + \chi \max\left\{\frac{\beta_0}{q} - \mu_0,\, 0\right\} \\
(\rho+\gamma)\mu_1 &= \alpha(1-\mu_1) + \chi \max\left\{\frac{\beta_1}{q} - \mu_1,\, 0\right\} \\
\rho\beta_0 &= \mu_0 + \varepsilon(\beta_1-\beta_0) + \chi \max\left\{q\,\mu_0 - \beta_0,\, 0\right\} \\
\rho\beta_1 &= \mu_1 + \alpha(\beta_0-\beta_1) + \chi \max\left\{q\,\mu_1 - \beta_1,\, 0\right\}
\end{aligned}
$$

Now, begin with assuming that $\chi = 0$. In that case, $\mu_0 < \mu_1 < 1$ is obvious given that $\gamma > -\rho$. Next, add the equations for β_0 and β_1 and arrange them to yield:

$$
(\rho+\alpha+\varepsilon)(\beta_1-\beta_0) = \mu_1 - \mu_0,
$$

establishing $\beta_0 < \beta_1$. Finally, the equations for β_0 and β_1 can easily be solved for in terms of μ_0 and μ_1, and then arranged to yield:

$$
\frac{\beta_0}{\mu_0} = \frac{\rho+\alpha+\varepsilon\mu_1/\mu_0}{\rho(\rho+\varepsilon+\alpha)} \qquad \text{and} \qquad \frac{\beta_1}{\mu_1} = \frac{\alpha\mu_0/\mu_1+\rho+\varepsilon}{\rho(\rho+\varepsilon+\alpha)}
$$

As $\mu_0 < \mu_1$, the claim that $\beta_0/\mu_0 > \beta_1/\mu_1$ follows, and the guess that households in state 0 (1) want to buy (sell) bonds has been verified for the case $\chi = 0$.

Now, suppose that there exists a $\chi > 0$ such that $\beta_0/\mu_0 < \beta_1/\mu_1$; in that case, by continuity there must also exist a $\bar{\chi} > 0$ such that $\beta_0/\mu_0 = \beta_1/\mu_1$ exactly. Then, we must have $q = \beta_0/\mu_0 = \beta_1/\mu_1$ as well, and all the trade surpluses are zero. But we have already established that when all trade surpluses are zero, the inequality $\beta_0/\mu_0 > \beta_1/\mu_1$ holds. So the existence of $\bar{\chi}$ is contradicted and the inequality must hold for any $\chi \in [0, \infty)$.

Now turn to the remaining two inequalities. Rearrange the equation for β_0:

$$\rho\beta_0 = \mu_0 + \varepsilon(\beta_1 - \beta_0) \qquad \Rightarrow \qquad \rho\frac{\beta_0}{\mu_0} = 1 + \varepsilon\frac{\beta_1 - \beta_0}{\mu_0}$$

This does not depend on χ. As $\beta_0 < \beta_1$ (proven above), the claim $\beta_0/\mu_0 > 1/\rho$ follows.

Finally, assume that $q = \beta_1/\mu_1$, consider the equation for β_1, and rearrange it:

$$\rho\beta_1 = \mu_1 + \alpha(\beta_0 - \beta_0) \qquad \Rightarrow \qquad \rho\frac{\beta_1}{\mu_1} = 1 - \alpha\frac{\beta_1 - \beta_0}{\mu_1}$$

As $\beta_0 < \beta_1$, the claim $\beta_1/\mu_1 < 1/\rho$ follows: when bonds are priced at the lowest level possible (i.e. when they are abundant), then their price reflects an illiquidity discount. □

Proof. Proof of Proposition 2. The strategy of the proof is simple: ignore the asset market clearing condition, let q be exogenous, and construct the rest of the equilibrium. Then, close the loop by solving for asset market clearing. What makes this work is the fact that all equations other than asset market clearing are linear in equilibrium variables other than ψ_0, ψ_1, and q (or determined in a block, such as h_0 and h_1). Furthermore, the costate equations (2) on the one hand, and the asset flow equations (7) together with goods market clearing and the government budget, form distinct blocks that we can solve separately.

First, costates. In steady state and taking q as given, the equations form a linear block:

$$\begin{aligned}
(\rho + \gamma + \varepsilon + \chi)\,\mu_0 &= \varepsilon\mu_1 + \frac{\chi}{q}\,\beta_0 \\
(\rho + \gamma + \alpha)\,\mu_1 &= \alpha \\
(\rho + \varepsilon)\,\beta_0 &= \mu_0 + \varepsilon\beta_1 \\
(\rho + \alpha + \chi)\,\beta_1 &= \mu_1 + \alpha\beta_0 + \chi\,q\,\mu_1
\end{aligned}$$

with a unique solution in terms of q. Now, the equilibrium requires that $\beta_1/\mu_1 \leq q \leq \beta_0/\mu_0$, so there are three possible outcomes: $q = \underline{q} = \beta_1/\mu_1$ (the lower bound), the interior, and $q = \overline{q} = \beta_0/\mu_0$ (the upper bound). Setting $q = \beta_1/\mu_1$, we can solve a quadratic equation and select the larger of two results (the only one guaranteed to be positive):

$$\begin{aligned}
\underline{q} &= \frac{\beta_1}{\mu_1} \\
&= \frac{1}{2\rho(\rho+\gamma+\varepsilon+\chi)(\rho+\varepsilon+\alpha)}\Big[\alpha\varepsilon + (\rho+\varepsilon)(\rho+\varepsilon+\gamma) + (2\rho+\alpha+\varepsilon)\chi \ldots \\
&\qquad + \sqrt{-4\rho\chi(\rho+\alpha+\varepsilon)(\rho+\gamma+\varepsilon+\chi) + [(\rho+\varepsilon)(\rho+\varepsilon+\gamma) + (2\rho+\varepsilon)\chi + (\varepsilon+\chi)\alpha]^2}\Big]
\end{aligned}$$

And setting $q = \beta_0/\mu_0$, we can solve for the upper bound, which is simply:

$$\overline{q} = \frac{\beta_0}{\mu_0} = \frac{(\rho+\gamma) + (\rho+\varepsilon+\alpha+\chi)}{\rho(\rho+\varepsilon+\alpha) - \gamma\chi},$$

if the denominator is positive. If on the other hand $\gamma\chi \geq \rho(\rho+\varepsilon+\alpha)$, then $\overline{q} = \infty$, and the equilibrium will never feature rationing of bond buyers.

Thus, what we need to do now is to solve for the rest of the equilibrium under three assumptions: that $q = \underline{q}$, $q \in (\underline{q}, \overline{q})$, and $q = \overline{q}$. Start in the interior:

Interior case: Suppose that $Z_0/B_1 \in (\underline{q}, \overline{q})$ (to be verified). Then, $\psi_0 = \psi_1 = 1$ and we can solve directly for steady-state bond holdings:

$$B_0 = \frac{\alpha+\chi}{\varepsilon+\alpha+\chi} B \qquad \text{and} \qquad B_1 = \frac{\varepsilon}{\varepsilon+\alpha+\chi} B \tag{A.3}$$

Next, take the flow equation for Z_0 in steady state and use the government's budget constraint to substitute T, to obtain:

$$\begin{aligned} & 0 = -\gamma Z_0 + B_0 + n_0(h_0 + \gamma(Z_0+Z_1) - B) - (\varepsilon+\chi) Z_0 \\ \leftrightarrow \qquad & Z_0 = \frac{1}{n_1\gamma+\varepsilon+\chi}\Big[B_0 - n_0 B + n_0 h_0 + n_0 \gamma Z_1\Big] \end{aligned}$$

And using goods market clearing to substitute for Z_1:

$$Z_0 = \frac{1}{n_1\gamma+\varepsilon+\chi}\Big[B_0 - n_0 B + n_0 h_0 + n_0 \frac{\gamma}{\alpha}(n_0 h_0 + n_1 h_1)\Big]$$

Finally, consider the candidate bond price Z_0/B_1:

$$\begin{aligned} \frac{Z_0}{B_1} &= \frac{1}{n_1\gamma+\varepsilon+\chi}\left[\frac{B_0}{B_1} - n_0 \cdot \frac{B}{B_1} + \frac{1}{B_1}\Big(n_0 h_0 + n_0 \frac{\gamma}{\alpha}(n_0 h_0 + n_1 h_1)\Big)\right] \\ &= \frac{1}{n_1\gamma+\varepsilon+\chi}\left[\frac{\alpha+\chi}{\varepsilon} - \frac{\alpha}{\varepsilon+\alpha} \cdot \frac{\varepsilon+\alpha+\chi}{\varepsilon} + \frac{\varepsilon+\alpha+\chi}{\varepsilon B}\Big(n_0 h_0 + n_0 \frac{\gamma}{\alpha}(n_0 h_0 + n_1 h_1)\Big)\right] \\ &= \frac{1}{n_1\gamma+\varepsilon+\chi}\left[\frac{\chi}{\varepsilon+\alpha} + \frac{\varepsilon+\alpha+\chi}{\varepsilon B}\Big(n_0 h_0 + n_0 \frac{\gamma}{\alpha}(n_0 h_0 + n_1 h_1)\Big)\right] \end{aligned} \tag{A.4}$$

after some algebraic simplification. This term only depends on exogenous variables, plus the labor supplies h_0 and h_1. But $h_1 = (v')^{-1}[\alpha/(\rho+\gamma+\alpha)]$ in steady state, which is also exogenous, and $h_0 = (v')^{-1}[\mu_0]$. Now, μ_0 is clearly decreasing in q (making bonds more expensive to buy discourages state-0 households from holding money that they could spend on bonds), and v' and its inverse are increasing functions, so the candidate price Z_0/B_1 must be decreasing in q. Consequently, there must be a unique crossing point that establishes the equilibrium – unless that crossing point lies outside the interval $(\underline{q}, \overline{q})$. We will consider these possibilities next.

Upper bound case: Suppose that $Z_0/B_1 > \overline{q}$. Then $\psi_1 = 1$ but $\psi_0 < 1$; in fact, we have:

$$\psi_0 Z_0 = \overline{q} B_1 = \overline{q}\, \frac{\varepsilon}{\varepsilon+\alpha+\chi} B$$

In this case, $\psi_1 = 1$ implies that the bond holdings in steady state still satisfy Equation (A.3). Z_1 is still determined by goods market clearing, and of course h_0 and h_1 are determined by the solution to the costates block at $q = \overline{q}$. The flow equation for Z_0 in steady state now becomes:

$$\begin{aligned} 0 &= -\gamma Z_0 + B_0 + n_0(h_0 + \gamma(Z_0 + Z_1) - B) - \varepsilon Z_0 - \frac{\chi\varepsilon}{\varepsilon+\alpha+\chi}\overline{q}B \\ \leftrightarrow \qquad Z_0 &= \frac{1}{n_1\gamma+\varepsilon}\Big[\frac{\alpha+\chi-\overline{q}\varepsilon}{\varepsilon+\alpha+\chi}\,B - \frac{\alpha}{\varepsilon+\alpha}B + n_0h_0 + n_0\gamma Z_1\Big] \end{aligned}$$

Again, using goods market clearing to substitute for Z_1:

$$Z_0 = \frac{1}{n_1\gamma+\varepsilon}\Big[\frac{\alpha+\chi-\overline{q}\varepsilon}{\varepsilon+\alpha+\chi}\,B - \frac{\alpha}{\varepsilon+\alpha}B + n_0h_0 + n_0\frac{\gamma}{\alpha}(n_0h_0+n_1h_1)\Big],$$

a unique solution. (If the implied result is $Z_0 \leq 0$, then it is simply not the case that the bond price is at the upper bound.)

Lower bound case: Suppose that $Z_0/B_1 < \underline{q}$. Then $\psi_0 = 1$ but $\psi_1 < 1$; in fact, we have:

$$\psi_1 B_1 = \frac{Z_0}{\underline{q}}$$

Substituting this into the bond flow equations, we get:

$$B_0 = \frac{\alpha B}{\varepsilon+\alpha} + \frac{\chi Z_0}{(\varepsilon+\alpha)\underline{q}} \qquad \text{and} \qquad B_1 = \frac{\varepsilon B}{\varepsilon+\alpha} - \frac{\chi Z_0}{(\varepsilon+\alpha)\underline{q}}$$

Finally, the flow equation for Z_0 in steady state yields, after the usual substitutions:

$$Z_0 = \frac{n_0h_0 + n_0\frac{\gamma}{\alpha}(n_0h_0+n_1hh_1)}{n_1\gamma+\varepsilon+\chi\left(1-\frac{1}{(\varepsilon+\alpha)\underline{q}}\right)}$$

Clearly, this term can blow up if the term $(\varepsilon+\alpha)\underline{q}$ is small enough. What does this mean? It means that if $(\varepsilon+\alpha)\underline{q}$ is small enough, then equilibrium is never in the upper bound region where bond sellers are rationed. For any positive bond supply, no matter how large, the equilibrium bond price is above the lower bound. Another way to see this is to go back to Equation (A.4) (the bond price $q = Z_0/B_1$ in the interior case), take the limit $B \to \infty$, and compare the result with $\underline{q}$. We obtain the result that bond sellers are never rationed if:

$$\frac{\chi}{(n_1\gamma+\varepsilon+\chi)(\varepsilon+\alpha)} > \underline{q}$$

However, this formula also clarifies that this is an odd region of the parameter space. χ needs to be large to make the left-hand side large, in which case $\underline{q} \approx 1/\rho$. The condition then reduces to $\varepsilon + \alpha < \rho$: state transitions need to be so rare as to be dominated by time preference. This is far away from any numerical experiment considered in this paper.

Sufficient conditions for a strongly-monetary steady-state equilibrium. So far, we have established that a unique solution to the steady-state equations exists. By definition, if this solution also satisfies $h_0 + T > 0$, then the solution describes a strongly-monetary steady-state equilibrium of the model. The condition is required for a tractable solution, because without it, some state-0 households would want to decumulate money balances rather than accumulate them. As decumulation makes the distance to zero relevant, it leads to value functions that are nonlinear in money holdings, rendering the model intractable. Unfortunately, there is no simple restriction on exogenous parameters equivalent to $h_0 + T > 0$, but there are some sufficient conditions.

For example, recall that the government budget constraint is $T = \gamma(Z_0 + Z_1) - B$. Therefore, we need $B < h_0 + \gamma(Z_0 + Z_1)$. Suppose $\gamma \geq 0$ so that there is no deflation that would have to be financed with tax revenue, then $B < h_0$ is sufficient. How large can we allow the debt service flow B (and the implied flow of lump-sum taxes T) to get? Fix all parameters other than B and T. Then the term h_0 achieves its minimum (call it $h_0^{\min}$) for $B \to 0$ and $q \to \beta_0/\mu_0$ (the inflation tax is most keenly felt when there are no saving vehicles other than money). In this case, $\mu_1 = \alpha/(\rho + \gamma + \alpha)$ and $\mu_0 = \varepsilon\alpha/(\rho + \gamma + \varepsilon)/(\rho + \gamma + \alpha)$. We can then use Equation (3) to find $h_0^{\min} \equiv (v')^{-1}[\varepsilon\alpha/(\rho + \gamma + \varepsilon)/(\rho + \gamma + \alpha)]$. One sufficient condition for existence of a strongly-monetary equilibrium is thus $B < h_0^{\min}$ together with $\gamma \geq 0$.

If instead $\gamma < 0$, so that there is deflation which must be financed with taxes, we must solve for Z_0 and Z_1, too, which can get complex even if a closed form solution exists. But suppose $\gamma \to -\rho$, so we are approaching the Friedman Rule in the limit and asset valuations become trivial: $\mu_0 = \mu_1 = 1$ and $\beta_0 = \beta_1 = 1/\rho$. Households work at the first-best level h^*, which is defined by $v'(h^*) = 1$.

Now, consider two subcases which we can solve completely: first, let the bond supply be zero, and second, let it be such that both buyers and sellers in the decentralized bond market always get to trade. That is, $\psi_0 = \psi_1 = 1$; notice that as we approach the Friedman Rule, the set of bond supplies for which equilibrium is in this intermediate region shrinks to a singleton.

In the first case, $B = 0$ implies $\psi_0 = 0$ and $\psi_1 = 1$, so we can ultimately solve for:

$$-\frac{T}{h^*} = \frac{\rho(\alpha^2 + \alpha\varepsilon + \varepsilon^2 - (\alpha + \varepsilon)\rho)}{\alpha\varepsilon(\alpha + \varepsilon - \rho)}$$

We want this term to be less than one; it is easy to verify that this is the case iff $\rho < \min\{\varepsilon, \alpha\}$.

And in the second case, where the bond supply is exactly such that both bond buyers and sellers trade with certainty ($\psi_0 = \psi_1 = 1$), we can solve for:

$$-\frac{T}{h^*} = \frac{\rho\left[2\alpha^3 + \alpha^2(2\varepsilon + 3\chi - 2\rho) + [\varepsilon\chi + \alpha(\varepsilon+\chi)](\varepsilon + \chi - 2\rho)\right]}{\alpha\left[\alpha\varepsilon(\alpha+\varepsilon-\rho) + \chi[(\alpha+\varepsilon)^2 - \varepsilon\rho] + \chi^2(\alpha+\varepsilon)\right]}$$

Again, we want this term to be less than one; one can verify that this is the case for $\rho < \min\{\varepsilon, \alpha, \chi\}$.

If none of these sufficient conditions apply, the condition $h_0 + T$ must be verified numerically for a candidate equilibrium. □

A.3 Extension: the asset market is intermediated by dealers

Assume that there is a measure χ of "dealers", special financial firms that intermediate asset trade. Households own a diversified portfolio of dealers' profits. Dealers have the ability to verify and certify bonds (which explains why households are unable to trade bonds directly, including paying for goods with bonds), and they are able to access a frictionless inter-dealer market (so they do not need to hold inventory). At rate χ, households are matched with dealers, and there is no direct asset trade between households. When matched, households and dealers determine the size of the trade and the split of the surplus by Nash bargaining, where the dealer has a bargaining power exponent of $\zeta \in [0, 1]$. (When $\zeta = 0$, the resulting equations are identical to those in the main text.)

First, consider a match between a household with real balances z and bonds b, and a dealer with access to the inter-dealer market where the price is q. Denote by b_M the quantity of bonds bought by the household (sold if negative), and by q_M the "match-specific price" charged or offered for those bonds. Guess ahead that the value function of money and bonds is again linear, with a marginal value of real balances μ and a marginal value of bonds β. Then the Nash bargaining solution satisfies:

$$(b_M, q_M) \in \underset{b_M, q_M}{\arg\max} \left\{(-\mu q_M b_M + \beta b_M)^{1-\zeta}(q_M b_M - q b_M)^{\zeta}\right\}$$
$$\text{subject to } b_M \in [-b, z/q_M]$$

The bargaining solution has two cases, corresponding to whether the household buys or sells bonds. The first case, where the household is a buyer, obtains if $\beta/\mu > q$:

$$b_M = z/q_M \quad \text{and} \quad q_M = \left[\zeta\left(\frac{\beta}{\mu}\right)^{-1} + (1-\zeta)q^{-1}\right]^{-1}$$

So the household will spend all of its money to buy bonds, and the ask price charged is the *harmonic mean* of the parties' marginal rates of substitution, weighted by their bargaining power. The second case, where the household is a seller, obtains if $\beta/\mu < q$:

$$b_M = -b \quad \text{and} \quad q_M = \zeta \frac{\beta}{\mu} + (1-\zeta) q$$

So the household will sell all of its bonds, and the bid price offered is the *arithmetic mean* of the parties' marginal rates of substitution, weighted by their bargaining power.

In equilibrium, because $\beta_0/\mu_0 > \beta_1/\mu_1$, households in state 0 will buy bonds at the ask price and households in state 1 will sell bonds at the bid price when matched with a dealer. Substituting the results into the households' Euler equations, these equations take the following form:

$$\rho\mu_0 = \dot{\mu}_0 - \pi\mu_0 + \varepsilon[\mu_1 - \mu_0] + \chi(1-\zeta)\left[\frac{\beta_0}{q} - \mu_0\right] \tag{A.5a}$$

$$\rho\mu_1 = \dot{\mu}_1 - \pi\mu_1 + \alpha[1 - \mu_1] \tag{A.5b}$$

$$\rho\beta_0 = \dot{\beta}_0 + \mu_0 + \varepsilon[\beta_1 - \beta_0] \tag{A.5c}$$

$$\rho\beta_1 = \dot{\beta}_1 + \mu_1 + \alpha[\beta_0 - \beta_1] + \chi(1-\zeta)\left[q\mu_1 - \beta_1\right] \tag{A.5d}$$

We can see that as far as households' decision making is concerned, giving dealers bargaining power $\zeta > 0$ is equivalent to increasing the trading delay (confirming the result of Lagos and Rocheteau, 2009). General equilibrium is a little bit more complicated now because dealer's profits need to be remitted to households, but the extension is straightforward and not much happens in terms of results. Since the main purpose of the model is understanding monetary intervention in illiquid markets, including dealers is not essential. It may matter for empirical purposes, however, or for extensions that take financial market microstructure more seriously.

A.4 Extension: bonds have direct liquidity

In the model from the main text, money is the only means of payment in the goods market; however, bonds can be traded for money on a market, which imbues them with *indirect liquidity* properties Geromichalos and Herrenbrueck (2016). Alternatively, or in addition, we can consider the *direct liquidity* paradigm (Geromichalos et al., 2007; Lagos and Rocheteau, 2008; Lester et al., 2011): bonds also serve as means of payment, but only with probability $\eta \in [0, 1]$ (and whether or not is revealed at the point of the purchase).[23]

Here, in the continuous-time environment with household heterogeneity, giving bonds direct liquidity properties is awkward and raises technical issues that distract from the research questions. Number one, can tax obligations be paid for with bonds? Number two, in the indirect liquidity model money is valued for any inflation rate (given that $v'(0) = 0$),

[23] So far, the only paper to consider both paradigms is by Geromichalos, Jung, Lee, and Carlos (2018), who find that secondary market liquidity of assets enhances their acceptability as a medium of exchange. A seller who has no use for an asset may still accept it, because she anticipates selling it on the secondary market.

so there is no need to keep track of an upper bound for inflation. This is not the case for the direct liquidity model, even if bonds are an inferior means of payment. Number three, with regards to bond trade in the asset market, there are *two* household types with distinct marginal rates of substitution (state-0 and state-1, β_0/μ_0 versus β_1/μ_1), so there is a natural direction of trade and the bond price must be between the two MRSs. However, in the goods market, there are *three* types of participants: workers in state 0, workers in state 1, and imminent consumers (transitioning from state 1 to state 0). Thus, there are three relevant MRSs when money is valued: β_0/μ_0 and β_1/μ_1 for workers in either state, and β_0 for imminent consumers (since their marginal valuation of real balances is 1 by construction). Plus, in case money is not valued, β_1 (the marginal value of bonds to state-1 workers). Since β_0 and β_1/μ_1 cannot be unambiguously ranked, and since the market price of bonds in the goods market must be between β_0/μ_0 (upper bound) and whichever of $(\beta_0, \beta_1/\mu_1)$ is larger (lower bound), but could be interior or at a corner, and since money might or might not be valued, the goods market clearing solution has five possible branches – excluding knife-edge cases. Denoting the amount of goods that one bond will buy by $\hat{q}$, these are:

- $\hat{q} = \beta_0/\mu_0$. State-0 workers accept both bonds and money as income, state-1 workers only take money
- $\beta_0/\mu_0 > \hat{q} > \max\{\beta_1/\mu_1, \beta_0\}$. State-0 workers accept only bonds, state-1 workers accept only money, and consumers spend all their money and (if allowed) bonds
- $\beta_0/\mu_0 > \hat{q} = \beta_0 > \beta_1/\mu_1$. State-0 workers accept only bonds, state-1 workers accept only money, and consumers spend all their money but only some of their bonds
- $\beta_0/\mu_0 > \hat{q} = \beta_1/\mu_1 > \beta_0$. All workers accept bonds, only state-1 workers accept money, and consumers spend all their money and (if allowed) bonds
- $\mu_0 = \mu_1 = 0$ (money is not valued), and all workers accept bonds at price $\hat{q} > \beta_1 > \beta_0$

Due to this complexity, a full general equilibrium analysis of the direct liquidity case is beyond the scope of this paper. Nevertheless, there are some things we can say for sure. First, we can show that trade in the frictional asset market must still flow in the natural direction even if bonds can be used as payment, as long as money is valued:

Proposition A.2. *Let $\eta < 1$, and money is valued ($\mu_0, \mu_1 > 0$). Then $\beta_0/\mu_0 > \beta_1/\mu_1$.*

Proof. For simplicity, assume that $\chi \approx 0$, so asset trade is rare. (The extension to $\chi > 0$ is a straightforward adaptation of the proof of Proposition 1 in Appendix A.2.) Then, the steady-state Euler equations are as follows:

$$\begin{aligned} (\rho+\gamma)\mu_0 &= \varepsilon(\mu_1-\mu_0) \qquad & \rho\beta_0 &= \mu_0 + \varepsilon(\beta_1-\beta_0) \\ (\rho+\gamma)\mu_1 &= \alpha(1-\mu_1) \qquad & \rho\beta_1 &= \mu_1 + \alpha(1-\eta)(\beta_0-\beta_1) + \alpha\eta\left(\max\left\{\hat{q},\beta_0\right\}-\beta_1\right), \end{aligned}$$

where the only difference from the main text is that bonds can be used to pay for goods with probability η, in which case one bond buys $\hat{q}$ of goods. Focus on the case where $\hat{q} > \beta_0$, otherwise consumers would simply hold on to their bonds voluntarily and direct bond liquidity becomes irrelevant.

We can divide the equations by μ_0 and μ_1, respectively, and rearrange them to get:

$$\mu_0 = \frac{\varepsilon}{\rho+\gamma+\varepsilon}\,\mu_1 \qquad\qquad \rho\frac{\beta_0}{\mu_0} = 1 + \varepsilon\,\frac{\mu_1}{\mu_0}\cdot\frac{\beta_1}{\mu_1} - \varepsilon\,\frac{\beta_0}{\mu_0}$$

$$\mu_1 = \frac{\alpha}{\rho+\gamma+\alpha} \qquad\qquad \rho\frac{\beta_1}{\mu_1} = 1 + \alpha\eta\,\frac{\hat{q}}{\mu_1} + \alpha(1-\eta)\,\frac{\mu_0}{\mu_1}\cdot\frac{\beta_0}{\mu_0} - \alpha\,\frac{\beta_1}{\mu_1}$$

Plug in for the fractions μ_1/μ_0 and $\hat{q}/\mu_1$, and rearrange the latter two equations to become:

$$(\rho+\varepsilon)\frac{\beta_0}{\mu_0} = 1 + (\rho+\gamma+\varepsilon)\frac{\beta_1}{\mu_1} \tag{A.6}$$

$$(\rho+\alpha)\frac{\beta_1}{\mu_1} = 1 + \eta(\rho+\gamma+\alpha)\hat{q} + (1-\eta)\frac{\varepsilon(\rho+\gamma+\alpha)}{\rho+\gamma+\varepsilon}\cdot\frac{\beta_0}{\mu_0} \tag{A.7}$$

Subtracting Equation (A.7) from (A.6):

$$(\rho+\varepsilon)\frac{\beta_0}{\mu_0} = (2\rho+\gamma+\varepsilon+\alpha)\frac{\beta_1}{\mu_1} - \eta(\rho+\gamma+\alpha)\hat{q} - (1-\eta)\frac{\varepsilon(\rho+\gamma+\alpha)}{\rho+\gamma+\varepsilon}\cdot\frac{\beta_0}{\mu_0}$$

Adding $(\rho+\alpha)\beta_0/\mu_0$ to both sides, and rearranging:

$$\begin{aligned}(2\rho+\gamma+\varepsilon+\alpha)\frac{\beta_0}{\mu_0} &= (2\rho+\gamma+\varepsilon+\alpha)\frac{\beta_1}{\mu_1} - \eta(\rho+\gamma+\alpha)\hat{q} \quad \dots \\ &\quad + \left(\rho+\alpha-(1-\eta)\frac{\varepsilon(\rho+\gamma+\alpha)}{\rho+\gamma+\varepsilon}\right)\frac{\beta_0}{\mu_0} \\ \leftrightarrow \qquad \frac{\beta_0}{\mu_0} &= \frac{\beta_1}{\mu_1} + \frac{\rho+\gamma+\alpha}{2\rho+\gamma+\varepsilon+\alpha}\left(\frac{\rho+\gamma+\eta\,\varepsilon}{\rho+\gamma+\varepsilon}\cdot\frac{\beta_0}{\mu_0} - \eta\,\hat{q}\right)\end{aligned}$$

Since $\beta_0/\mu_0 \geq \hat{q}$ (otherwise no worker would accept bonds in exchange for goods) and $(\rho+\gamma+\eta\,\varepsilon)/(\rho+\gamma+\varepsilon) > \eta$, the term in parentheses is positive. This proves the claim. □

Thus, trade in the bond market always flows in the natural direction: money is directed into the hands of those who need it more. Since most of the results of the paper can be traced back to this fact, they do not depend on the assumption that money is the only medium of exchange – they only require that money is the *best* medium of exchange.

Another thing we can do with the direct liquidity extension is to analyze how the steady-state asset valuations would change for the three special cases of the goods-market bond price $\hat{q}$. Including asset trade, the steady-state Euler equations are as follows:

$$
\begin{aligned}
(\rho+\gamma)\mu_0 &= \varepsilon(\mu_1-\mu_0)+\chi\left(\frac{\beta_0}{q}-\mu_0\right)\\
(\rho+\gamma)\mu_1 &= \alpha(1-\mu_1)\\
\rho\beta_0 &= \mu_0+\varepsilon(\beta_1-\beta_0)\\
\rho\beta_1 &= \mu_1+\alpha(1-\eta)(\beta_0-\beta_1)+\alpha\eta(\hat{q}-\beta_1)+\chi\left(q\,\mu_1-\beta_1\right)
\end{aligned}
$$

Accordingly:

- Let $\hat{q}=\beta_0$, which is obtained if ηB_1 is large (consumers hold more bonds than they can spend). Clearly, the Euler equations are exactly the same as in the main text.
- Let $\hat{q}=\beta_1/\mu_1>\beta_0$, which is obtained if ηB_1 is small (consumers spend all their bonds) but not too small ($\alpha\eta B_1>n_0h_0$, so state-0 workers do not take money and even state-1 workers take some bonds as income). In this case, the flow bond value $\rho\beta_1$ gets a small boost, equal to $\alpha\eta(\beta_1/\mu_1-\beta_0)$.
- Let $\hat{q}=\beta_0/\mu_0$, which is obtained if $\alpha\eta B_1<n_0h_0$ (even state-0 workers take some money as income). In this case, the flow bond value $\rho\beta_1$ gets a larger boost, equal to $\alpha\eta(1-\mu_0)\beta_0/\mu_0$. However, this boost will never be big enough to reverse the direction of asset trade, as shown in Proposition A.2 above.

Since asset trade still moves bonds away from state-1 households and money towards them, consumers in this model will actually not have that many bonds to use as payment instruments as long as $\chi\psi_1$ is large enough. Thus, it stands to reason that the general equilibrium results are not strongly affected by a small amount of direct liquidity ($\eta>0$ but not large) as long as bonds are scarce enough to be priced at a premium (which is the interesting case anyway) and χ is large enough (which is realistic; see the calibration in the Web Appendix).

A.5 Comparative statics of inflation and trading frequency

The discussion in Section 3.3 has already suggested that the comparative statics of inflation are important, not least because they are entwined with the comparative statics of bond supply under fiscal dominance. Focus on the baseline model without capital, and begin with solving the household's steady-state Euler equations for a given bond price q. We learn that $\beta_0/\mu_0>\beta_1/\mu_1$ (relative to money, households in state 0 value bonds more than households in state 1; see Proposition 1), and as a function of q, both β_0/μ_0 and β_1/μ_1 are increasing. Consequently, the highest possible price of bonds in terms of real balances, given all other parameters affecting asset valuations, is obtained by solving the Euler equations (2) jointly with $q=\beta_0/\mu_0$; denote this price by $\overline{q}$. Similarly, the lowest possible price of bonds is obtained by solving the Euler equations jointly with $q=\beta_1/\mu_1$; denote this price by $\underline{q}$.

Proposition A.3. *For a given set of parameters $(\rho,\varepsilon,\alpha,\chi)$, the bond price bounds satisfy:*

1. *If $\pi > -\rho$ (away from the Friedman rule), then $\underline{q} < 1/\rho < \overline{q}$;*

2. *As $\pi \to -\rho$ (near the Friedman rule), $\underline{q} \to 1/\rho$ and $\overline{q} \to 1/\rho$;*

3. *As π increases, $\underline{q}$ strictly decreases and $\overline{q}$ strictly increases;*

4. *$\overline{q} < \infty$ if and only if $\pi\chi < \rho(\rho + \varepsilon + \alpha)$.*

Proof. Setting $q = \beta_1/\mu_1$, and again beginning with $\chi = 0$, we can solve:

$$\underline{q} = \frac{\beta_1}{\mu_1} = \frac{(\rho + \pi + \varepsilon)(\rho + \varepsilon + \alpha) - (\rho + \pi)\alpha}{\rho(\rho + \pi + \varepsilon)(\rho + \varepsilon + \alpha)} < \frac{1}{\rho}$$

As $\chi \to \infty$, we have $\beta_1/\mu_1 \to 1/\rho$; so the claim holds for every $\chi \in [0, \infty)$.

Now setting $q = \beta_0/\mu_0$, we can solve:

$$\overline{q} = \frac{\beta_0}{\mu_0} = \frac{(\rho + \pi) + (\rho + \varepsilon + \alpha + \chi)}{\rho(\rho + \varepsilon + \alpha) - \pi\chi}$$

Clearly, $\overline{q} > 1/\rho$, and $\overline{q} < \infty$ if and only if $\pi\chi < \rho(\rho + \varepsilon + \alpha)$.

And as $\pi \to -\rho$, we must have $\mu_0 \to \mu_1$, hence $\beta_0 \to \beta_1$, and consequently, $\underline{q} \to 1/\rho$ and $\overline{q} \to 1/\rho$ for any χ. □

So in principle, as inflation increases, the market price for bonds could increase or decrease as the range of acceptable terms of trade widens. But it is Z_0/B_1, the ratio of bond demand to supply, that determines whether the price is at a boundary or in the interior. As inflation does not affect the steady-state value of B_1 in the interior region where $\psi_1 = 1$, it is the effect of inflation on the real balances held by households in state 0 that determines the behavior of the bond price. As it turns out, Z_0 is either strictly decreasing or hump-shaped as a function of inflation. Starting at the Friedman rule $\pi = -\rho$, a small increase in π has two effects: it reduces the rate at which households in state 0 accumulate money, but through the inflation tax, it redistributes money from those who hold more on average (households in state 1) to those who hold less on average (households in state 0). Either effect could be stronger. But as π becomes very large, all households hold few real balances, so Z_0 necessarily tends to zero.

The fact that $\overline{q} \to \infty$ for a finite inflation rate means that, in an equilibrium in which buyers of bonds are rationed, potential buyers (households in state 0) are willing to hold real bonds with a zero rate of return, because avoiding the inflation tax is such a valuable service. As bond buyers would be willing to pay any price in the frictional asset market, they could never be rationed. Therefore, the region where bond buyers are rationed does not exist in this case, and the demand curve for bonds asymptotically approaches the vertical axis.

As Figure A.1 shows, the level of bond supply plays an essential role in determining the comparative statics of inflation. When B is low (bottom row), then the price of bonds

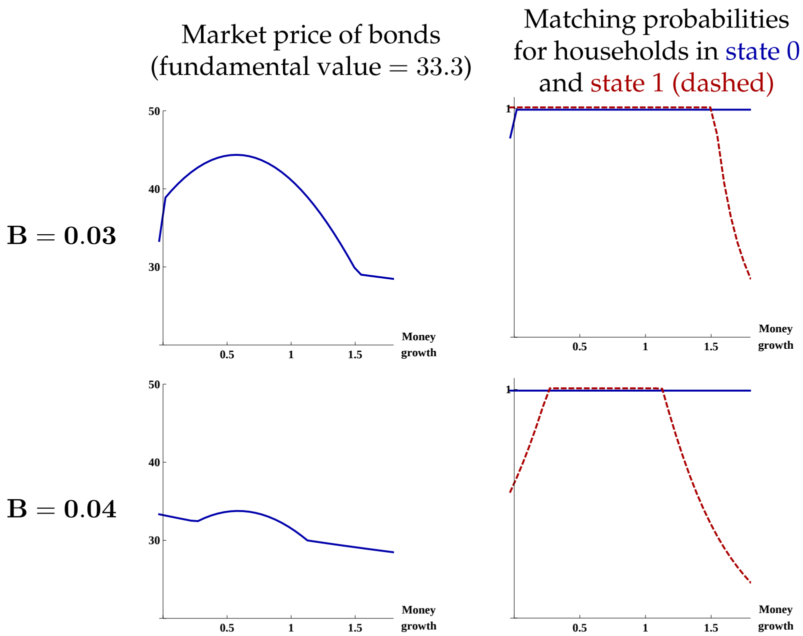

Figure A.1: Comparative statics of inflation. Key parameters: $\rho = .03$, $\varepsilon = .5$, $\alpha = 1$, $\chi = 6$.

tends to be at the lower bound $\underline{q}$, which is decreasing in inflation. The hump-shaped effect of inflation on bond demand, however, may make the price of bonds non-monotonic, too. When B is high (top row), by contrast, the price of bonds can be at the upper bound $\overline{q}$, which is increasing in inflation. As Z_0 cannot become arbitrarily large, the bond price is necessarily non-monotonic in inflation in this case. However, as the figure also makes clear, the non-monotonic effects of inflation likely require very high rates of inflation, on the same order of magnitude as the arrival rates of the state transitions (i.e., the liquidity shocks).

The comparative statics of bond liquidity, represented by the rate of matching in the decentralized asset market χ, can be described as follows. If both inflation and bond liquidity are so high that $\chi\pi \geq \rho(\rho + \varepsilon + \alpha)$, then $\overline{q}$ is infinite. The region where bond buyers are rationed does not exist, and the demand curve for bonds asymptotically approaches the vertical axis. If $\chi\pi < \rho(\rho + \varepsilon + \alpha)$, by contrast, then $\overline{q} < \infty$, so the price of bonds is bounded from above. The bond price bounds $\overline{q}$ and $\underline{q}$ are increasing in χ, so if either side of the market is rationed in equilibrium, then an increase in the rate of matching increases the price of bonds. The same is not necessarily true in the interior region, however; Z_0 and B_1 are both equilibrium objects, and their ratio may increase or decrease as the rate of matching increases. Put another way, an increase in bond liquidity will reduce yields if the bonds are either abundant or scarce, but anything can happen in the interior region.

Regarding the full model with bonds and capital, a numerical illustration of comparative statics with respect to the trading frequencies χ^B and χ^K can be found in the Web Appendix.