Quantitative Easing and the Liquidity Channel of Monetary Policy

Lucas Herrenbrueck
Simon Fraser University

This version: December 2016

Abstract

How do central bank purchases of illiquid assets affect interest rates and the real economy? In order to answer this question, I construct a flexible general equilibrium model of asset liquidity. In the model, households are heterogeneous in their asset portfolios and demand for liquidity, and asset trade is subject to frictions. I find that open market purchases of illiquid assets are fundamentally different from helicopter drops: asset purchases stimulate private demand for consumption goods at the expense of demand for assets and investment goods, while helicopter drops do the reverse. A temporary program of quantitative easing can therefore cause a 'hangover' of elevated yields and depressed investment after it has ended. When assets are already scarce, further purchases can crowd out the private flow of funds and cause high real yields and disinflation, resembling a liquidity trap. In the long term, lowering the stock of government debt reduces the supply of liquidity but increases the capital-output ratio. The consequences for output are ambiguous in theory but a calibration to US data suggests that the liquidity effect dominates; in other words, the supply of Treasuries is too small.

JEL codes: E31, E40, E50, G12

Keywords: Monetary theory, asset liquidity, indirect liquidity, search frictions, quantitative easing, liquidity trap

Contact: herrenbrueck@sfu.ca, +1-778-782-4805
Department of Economics, 8888 University Drive, Burnaby, B.C. V5A 1S6, Canada

1 Introduction

The recent quantitative easing programs in Japan, the United States, and the Eurozone have renewed theoretical interest in the question of how monetary policy can affect long-term interest rates, borrowing costs, and the real economy. With short-term rates at zero, central banks hope to gain traction with purchases of illiquid assets, such as long-term government bonds, federal agency debt, and privately-issued mortgage-backed securities. The empirical literature analyzing recent versions of quantitative easing suggests that the purchases were effective in reducing yields, but due to the lack of a suitable counterfactual, measuring the effects on the broader economy is very difficult, if at all possible. Consequently, understanding how the prices of illiquid assets can be related to each other and to the quantities in supply and demand, and how government intervention can affect these relationships, remains a priority for macroeconomic theory.

For this purpose, I construct a general equilibrium model of a production economy with heterogeneous households and multiple assets. Households receive random opportunities to purchase goods with money, and they are heterogeneous in how soon they expect these opportunities to arrive. This simple set-up is enough to make households differ in how much they value money and other financial assets, and therefore gives them a motive to trade assets with one another in financial markets. Monetary policy can be modeled either as intervention in these financial markets or as direct interaction with households' budget constraints (helicopter drops). The model shows that contrary to conventional wisdom, these two types of intervention have different (and in some ways opposite) effects, so this distinction is very important for predicting the effects of a new policy.

The main version of the model includes three assets: fiat money, a long-term bond issued by the government, and physical capital produced by private agents. Assets other than money are illiquid in the sense that they cannot be traded instantly, but are traded in frictional asset markets with trading delays and bid-ask spreads. However, these assets do obtain endogenous 'moneyness' because they can be *liquidated*, i.e., traded for money, by households who value money highly. As a consequence, and in contrast to standard asset pricing theory, bonds and capital are not only valued for their dividend streams, but also for how easily they can be liquidated, and at what price.

The chief result of the paper is that open market purchases of illiquid assets have both a direct effect on yields and an indirect portfolio balance effect. First, the demand curves of bonds and capital are downward sloping, giving scope to monetary policy to affect their prices; and these assets are imperfect substitutes even if they are traded in segmented financial markets, therefore a policy which reduces the supply of bonds will increase the price and quantity of capital. Second, the purchases reallocate portfolios among agents in the economy, directing money in a specific direction: towards agents who were seeking to sell assets and away from agents who were seeking to purchase them. If those agents who anticipate good consumption opportunities are also the ones

most likely to liquidate financial assets in order to obtain money, then they will be over-represented among asset sellers; this is the case in the model and it seems reasonable in reality, too. As open-market purchases by their nature redistribute purchasing power towards asset sellers, they stimulate the demand for new consumption and investment goods.

The consequences for quantitative easing can be summarized as follows. Temporary open-market purchases of long-term government bonds tend to reduce the yields on these bonds and, indirectly, on other assets such as physical capital, and can thereby stimulate capital accumulation and output. Whether this effect has quantitative power will depend on a number of factors, such as the degree of asset market integration, the elasticity of investment with respect to the price of capital, and the wage elasticity of the labor supply. But on the whole, the conclusion is that quantitative easing can work, and in fact do so through the same channels as 'standard' monetary policy. However, the model also suggests three new reasons why quantitative easing may fail.

First, one of the reasons the program works is because the purchases direct money more quickly into the hands of households likely to spend it on new goods and services, and out of the hands of households seeking to save, i.e. spend money on assets. When the program ends, private demand for financial assets will therefore be depressed below the long-run equilibrium level, crowded out by excess demand for assets by the central bank, leading to a "hangover" of higher interest rates and slower investment after the stimulus is withdrawn.

Second, non-monetary assets have a positive rate of return and therefore help households store their wealth more efficiently than in the form of money. Those households who do not expect to need liquid money soon are particularly sensitive to this. Reducing the supply of government bonds has therefore a long-term economic cost which has to be balanced against any gains from increased capital accumulation.[1] In fact, in a calibration of the model to US data, I find that this cost from a reduced supply of government bonds is likely to outweigh the increase in capital intensity, leading to lower output in the long run.[2]

Third, the intervention will also affect the flow of assets between households. These asset flows matter in ways that a representative household model cannot capture. For example, the model features a case where due to fundamentals (preferences and market structure), asset prices are inelastic to asset supply at an elevated level. In such a case, open market operations can be ineffective or even counterproductive: households will hold on to any additional liquidity, reducing the velocity of circulation and, consequently, medium-run expectations of the price level. This will cause at least temporary disinflation, but it is also possible that the lower velocity of circulation (in textbook terms, an increase in money demand) soaks up future increases in money supply and may

[1] Williamson (2012) identified the lower long-run supply of liquid assets as the main cost of quantitative easing policies, but did not study the possible gains from capital accumulation.

[2] Though in reality, a lower quantity of government debt would additionally reduce the costs of distortionary taxation. This concern is absent from the model in which taxes are lump-sum.

increase real interest rates, reduce capital accumulation, and contract the economy.[3] This result is especially relevant to the current policy discussion because it resembles the original conception of a "liquidity trap" as a region where the relative demand of bonds and money is flat (Robertson, 1940); having been derived in a model where bonds are real and prices are perfectly flexible and determined in competitive spot markets, it strongly suggests that the existence of a liquidity trap is not tied to price stickiness or the zero lower bound on nominal interest rates.

While there was a lack of evidence when the policy discussion around quantitative easing started, a growing body of empirical evidence now supports the contention that asset quantities do affect yields directly. D'Amico and King (2013), Gagnon, Raskin, Remache, and Sack (2011), and Bauer and Neely (2014) find that the early rounds of asset purchases in 2008-10 reduced yields, certainly for the assets purchased, and also for some assets that were not directly targeted (although Thornton (2012) disagrees). Krishnamurthy and Vissing-Jorgensen (2013) suggest that the purchases of 2008-10 had modest effects on yields, and that the evidence is mixed on the yields of assets not purchased under the program. Taking a broader approach, Krishnamurthy and Vissing-Jorgensen (2012) and Greenwood and Vayanos (2014) find evidence that the total supply of US government bonds of a given maturity mattered for yields even before 2007, a clear demonstration that as predicted by liquidity-augmented theories of asset pricing, asset demand curves do slope down and portfolio effects exist.

The argument that asset market frictions are the source of monetary non-neutrality and make intervention effective has a long tradition in monetary theory (Baumol, 1952; Tobin, 1956). An more recent incarnation is the "limited participation" literature, in which not all agents participate in asset markets, and some agents face cash-in-advance or borrowing constraints (Fuerst, 1992; Alvarez, Atkeson, and Kehoe, 2002; Williamson, 2006). Even more recently, Del Negro et al. (2011) and He and Krishnamurthy (2013) have used the fact that capital can serve as collateral for borrowing or credit to study how policy can stimulate capital accumulation. Gertler and Karadi (2011) focus on balance sheet constraints. The basic mechanism in Cúrdia and Woodford (2011) has in common with my paper that households are heterogeneous and differ in their demand for liquidity; they model it as patience shocks that make some households want to borrow, whereas I model it as differences in how soon random opportunities to spend money are likely to arrive.

My model is a hybrid of a monetary-search model in the tradition of Lagos and Wright (2005) and a model of frictional asset markets in the tradition of Duffie, Gârleanu, and Pedersen (2005), Lagos and Rocheteau (2009), and Trejos and Wright (2011).[4] The literature which uses search theory to study monetary policy and asset prices is extensive; Geromichalos, Licari, and Suárez-

[3] Most monetary models assume that seigniorage revenue is kept proportional to the money supply, generating a constant and exogenous rate of money growth. An equally reasonable assumption would be that seigniorage revenue is fixed in real terms. This difference matters a great deal.

[4] Williamson and Wright (2010) and Nosal and Rocheteau (2011) provide excellent surveys.

Lledó (2007), Berentsen, Camera, and Waller (2007), Lagos (2010, 2011), Berentsen and Waller (2011), and Rocheteau and Wright (2012) are prominent milestones. Here, I study the pricing of a real asset that cannot be used in exchange but has endogenous liquidity properties because it can be traded for money in a frictional asset market, as do Geromichalos and Herrenbrueck (2016), Lagos and Zhang (2015), Berentsen, Huber, and Marchesiani (2014), Mattesini and Nosal (2015), Huber and Kim (2015), and Herrenbrueck and Geromichalos (2015).[5]

In addition to asset market frictions, the second key feature of the model is household heterogeneity with respect to the (endogenous) demand for money. Prominent monetary-search models that have studied portfolio heterogeneity include Berentsen, Camera, and Waller (2005), Chiu and Molico (2010, 2011), and Rocheteau, Weill, and Wong (2015). All of them study the distribution of money holdings arising from idiosyncratic trading history, and diminishing marginal utility of money implies that interventions which compress the distribution of money holdings (e.g., helicopter drops) can be welfare-enhancing. This is in stark contrast to the model here, where households hold more money on average *if they expect to need it soon*. As a consequence, compressing the distribution of real money holdings will reduce the demand for goods, not increase it.

As I study the effect of monetary policy on the accumulation of physical capital, my paper is also part of a literature going back to Tobin (1965). Recent examples include papers by Lagos and Rocheteau (2008), Rocheteau and Rodriguez-Lopez (2014), and Aruoba, Waller, and Wright (2011). In the former three papers, anticipated inflation generally leads to a higher capital stock (potentially to the point of overaccumulation), but the latter paper finds the opposite. My model nests both outcomes. If capital is scarce relative to other assets, then households value it relatively highly for its liquidity properties, and if in addition the labor supply is inelastic, then moderate inflation increases capital accumulation and output. If, on the other hand, capital is relatively abundant so that (at the margin) it is not valued for liquidity, and the labor supply is elastic with respect to the marginal utility of wealth, then inflation always reduces the capital stock and output.

The rest of the paper is organized as follows. The description of the model is split in two sections. In Section 2, I describe a baseline version of the model in order facilitate understanding of its core mechanisms. In Section 3, I add to the model investment, capital accumulation, and government intervention in asset markets; physical capital serves a dual role as an input in production and a saving vehicle traded in asset markets, and government intervention can affect the aggregate stock of capital. Section 4 describes the calibration of the model, and Section 5 concludes.

[5] Applying models of asset market frictions to markets of government bonds is sometimes challenged because these markets are considered highly liquid, especially in the case of the US. However, the frictional model is valid as long as the markets are not *perfectly* liquid, and of course no real-world market is. For example, Ashcraft and Duffie (2007) documented the relevance of frictions in the federal funds market, which at the time was thought of as one of the most liquid markets in existence. Furthermore, if Treasuries were exactly as liquid as cash they could not be priced at a positive nominal yield by agents who also held money; but they are (with notably rare exceptions).

2 A Model of Asset Liquidity

The model is based on Rocheteau, Weill, and Wong (2015) in its description of the monetary environment and the structure of goods and labor markets. There are three innovations. First, households are heterogeneous in how soon they expect to need money. Second, there are financial assets in addition to money. Third, these assets can be traded in frictional asset markets à la Duffie, Gârleanu, and Pedersen (2005). Because households are heterogeneous, there exist gains from trade in that some households would like to sell assets for money and others would like to buy them.

The full model with both government bonds and physical capital as competing assets, and with government intervention in asset markets, is fairly complex. As a result, the exposition will proceed in steps building up from the basic environment to the more complex details later. For now, there are only two assets (money and real government bonds), and the government can interact with the households' budget constraints but not with goods or asset markets.

2.1 Environment

Time $t \in [0, \infty)$ is continuous and goes on forever. There are four types of agents: households, good-producing firms, financial brokers, and a government. Households have unit measure and are infinitely lived. Firms and brokers make zero profits at any time, so their measure and lifetime is indeterminate. The government is a single consolidated authority that can create assets, make transfers, and collect taxes.

There are five commodities in the model. The first is a flow consumption good, called "fruit", and denoted by c. It will serve as the numéraire in this economy. The second commodity is a lumpy consumption good, denoted by d, which can only be consumed as a stock at certain random opportunities. The third is labor effort, denoted by h, which is expended as a flow. All of the first three commodities are perishable and generate utility. The final two commodities are assets: they are perfectly durable and do not generate utility. First, there is a real consol bond b, which pays a flow dividend of one unit of numéraire (and never matures). The final commodity is fiat money, denoted by m, which pays no dividend.

The supply side of the economy is easily described. Each household owns $\bar{h} < \infty$ units of labor. Firms can transform labor h into fruit c or the lumpy consumption good d at a constant marginal cost of 1. The supplies of bonds and money, $B(t)$ and $M(t)$, are controlled by the government.

Households are ex-ante identical but can be in one of two states, 0 and 1, distinguished by how likely the random opportunity to consume the stock good d is. In state 0, households never receive such opportunities, but they may transition to state 1 at Poisson arrival rate $\varepsilon > 0$. In state 1, households receive opportunities to consume the lumpy good d at Poisson arrival rate $\alpha > 0$.

Immediately after such shocks, they transition back into state 0, an assumption which is made without loss of generality.[6] Figure 1 provides an illustration.

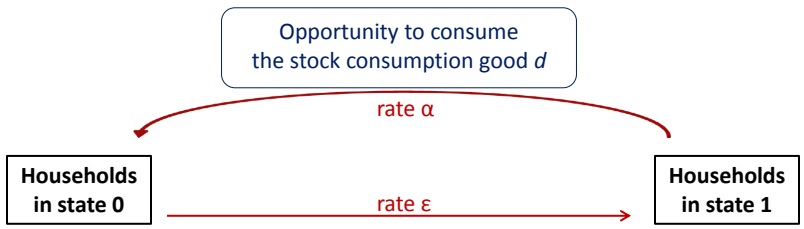


Figure 1: Households can be in one of two states

These shocks could be interpreted in two ways. First, the household may simply want to consume good d at exactly that instant and at no other (a taste shock that arrives in two stages). Second, the household may always desire to consume good d while in state 1, but the retail market for that good is decentralized and subject to search-and-matching frictions, and matches between firms and households in state 1 are generated at Poisson rate α (one taste shock and one matching shock); this second interpretation will be used throughout the paper.

Households are anonymous in the retail market, and therefore credit arrangements are not feasible because households would renege on any promise. Add to this the fact that labor and fruit are not storable, and an infinite supply of labor at an instant in time is physically impossible, then it follows that households wishing to consume good d must pay for it with some sort of liquid asset. Consequently, we may interpret state 0 as the "low demand for liquidity" state and state 1 as the "high demand for liquidity" state.

Households discount time at rate $\rho > 0$. Fruit consumption c and labor effort h generate flow utility $u(c,-h)$, and consumption of d units of the lumpy good (at random time T_1) generates utility d: the marginal utility of the lumpy good is constant and normalized to 1. As a result, we can write the utility of a household in state 0, $U_0(t)$, and the utility of a household in state 1, $U_1(t)$, in the recursive form:

$$U_0(t) = \mathbb{E}\left\{\int_t^{T_0}\left[e^{-\rho(\tau-t)}u(c(\tau),-h(\tau))\right]\mathrm{d}\tau + e^{-\rho(T_0-t)}U_1(T_0)\right\}$$
$$U_1(t) = \mathbb{E}\left\{\int_t^{T_1}\left[e^{-\rho(\tau-t)}u(c(\tau),-h(\tau))\right]\mathrm{d}\tau + e^{-\rho(T_1-t)}\left(d(T_1)+U_0(T_1)\right)\right\}$$

[6] Getting ahead of the story: the value functions will be linear and the marginal rates of substitution between money and other assets will not depend on a household's portfolio, merely on its state. Households in state 1 are the only ones with a chance to use money, therefore they will value it more. If they expected to stay in state 1 after making the purchase, they would still value money more than households in state 0.

where the first expectation is over the random time T_0 (which arrives at rate ε), and the second expectation is over the random time T_1 (which arrives at rate α).

The function u is strictly increasing and strictly concave in each argument. Furthermore, assume that $\bar{c}$ exists such that $u_1(\bar{c},-\bar{c}) = u_2(\bar{c},-\bar{c})$ (interpreted as the maximal fruit consumption of a household who never saves for consumption of the lumpy good). Finally, assume that $u_1(\bar{c},-\bar{c}) < 1$ (given a suitable medium of exchange, households do want to save for consumption of the lumpy good) and that $u_2(c,-\bar{h}) > 1$ for every $c \in [0,\bar{c}]$; this implies that $\bar{c} \in [0,\bar{h})$ and the constraint $h \leq \bar{h}$ never binds.

Both money and real bonds are durable and perfectly divisible, but only money is recognizable by everyone in this economy. Firms cannot recognize bonds, therefore they will not accept them as medium of exchange in any trades.[7] The function of financial brokers is that only they can verify and certify the authenticity of bonds, and are therefore able to serve as intermediaries to households wishing to trade bonds for money.

2.2 Market structure

There exists an integrated competitive spot market which is always open, in which households and firms trade labor, money, and the numéraire consumption good. Furthermore, there is a decentralized goods market where households in state 1 are matched with firms at Poisson rate α for an opportunity to buy the lumpy consumption good. As explained above, money is the only possible means of payment in this market. To keep this market simple, I assume that the household makes the firm a take-it-or-leave-it offer, equivalent to competitive pricing in this context.

Because the marginal rate of transformation of labor into goods is 1, and because firms make no profits, labor market clearing implies that the wage is 1 unit of fruit per unit of labor at any time. Denote the price of money in terms of fruit by ϕ, and express any money holdings m as real balances $z \equiv \phi m$ (so we can describe equilibrium in terms of stationary variables only). The inflation rate is $\pi \equiv -\dot{\phi}/\phi$; an increase in the price of goods is a fall in the price of money, and a household holding a constant stock of money expects its purchasing power to decay at rate π.

There is also a decentralized asset market where households in either state are matched with brokers at Poisson rate χ, and they may exchange any combination of money or bonds at market price q (measured in terms of real balances per bond).[8] There is no inter-household asset trade, and households make the broker a take-it-or-leave-it offer. Finally, brokers have access to a com-

[7] Nosal and Wallace (2007), Rocheteau (2009), and Lester et al. (2011) establish that money can emerge as a unique medium of exchange if it is at least somewhat more trustworthy than other assets. Li and Rocheteau (2011) and Rocheteau (2011) provide conditions under which assets are still accepted in trade even if they can be counterfeited.

[8] Introducing additional perfectly liquid assets, such as demand deposits, would not affect the analysis much; they would behave as perfect substitutes to money.

petitive inter-dealer asset market in order to fulfill their clients' orders, so they never need to hold inventory.[9]

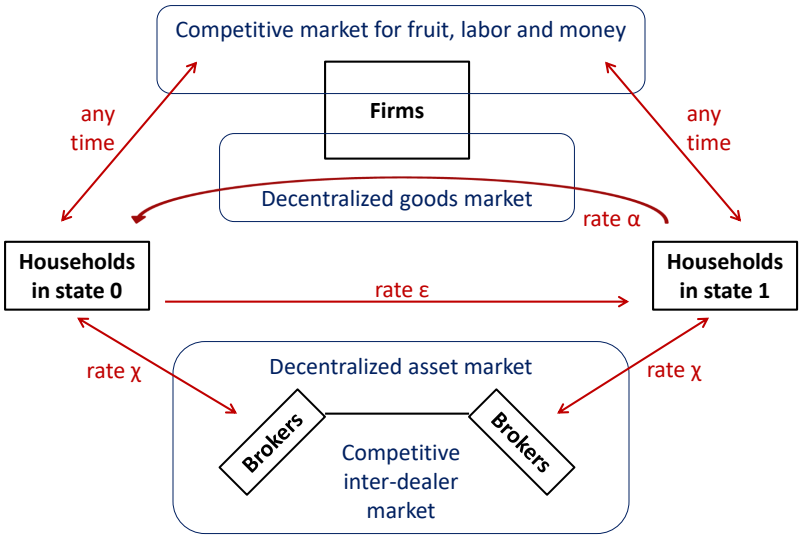


Figure 2: Illustration of the market structure sans government intervention

The government can make lump-sum transfers T of real balances to households (or collect taxes if $T < 0$). They are lump-sum in terms of applying equally to all households. But it is important to keep in mind that they are being assessed as *flows*, i.e. they affect the *rate of change* of households' money holdings and not the holdings directly. The government has to service its debt by paying a flow dividend of one unit of real balances to the owner of one unit of bonds. For the baseline version of the model, I assume that the supply of bonds is exogenous and fixed over time, but in Section 3, I describe how the government can issue bonds, retire them, and intervene in the frictional asset market.

2.3 Household's problem

Households decide on the flow of fruit consumption $c(t)$, on the flow of labor effort $h(t)$, how many real balances $z(t)$ to accumulate, and how much to trade in decentralized meetings. Recall that the real wage is 1, and by the definition of real balances the price of both consumption goods in terms of real balances is also 1. When given a random opportunity to consume the lumpy consumption good, a household with z real balances chooses to purchase $d(z) \in [0, z]$ units of the good. When

[9] Having brokers with no market power makes the model mathematically equivalent to having a competitive bond market that can only be accessed with a random delay. But the existence of brokers is theoretically helpful because it allows asset markets to be segmented. *Agents* might be able to recognize all assets, money, bonds, and capital, but *brokers* are defined by recognizing only money plus one other asset that they are specialized in. Furthermore, appendix C describes an extension of the model where brokers have some bargaining power.

matched with a broker, a household in state 0 with z real balances and b bonds chooses to buy $s_0(z,b) \in [-b, z/q]$ units of bonds, at the prevailing market price q because the broker has no bargaining power, and a household in state 1 chooses to sell $s_1(z,b) \in [-z/q, b]$ units of bonds.[10]

Let $W_0(z,b)$ be the value function of an unmatched household in state 0 and $W_1(z,b)$ be the value function of an unmatched household in state 1. Now consider a household in state 1 who was matched with a firm. Because the marginal labor cost of producing either the numéraire good c or the lumpy good d is 1, firms are willing to produce the lumpy good at real price 1. As households make a take-it-or-leave-it offer to the firm, their value of being in the match can then be written as:

$$V(z,b) = \max_{d \in [0,z]} \left\{ d + W_0(z-d, b) \right\} \tag{1}$$

The constraint $d \leq z$ represents the fact that real balances are the only feasible medium of exchange.

The value to a household of being matched with a broker can be written as:

$$\Omega_0(z,b) = \max_{s_0 \in [-b, z/q]} \left\{ W_0(z - qs_0, b + s_0) \right\} \tag{2a}$$

$$\Omega_1(z,b) = \max_{s_1 \in [-z/q, b]} \left\{ W_1(z + qs_1, b - s_1) \right\} \tag{2b}$$

where q is the inter-dealer market price of bonds in terms of real balances. (The household has all the bargaining power in the match, and trading at that price maximizes the household's surplus.)

The value functions $W_0(z,b)$ and $W_1(z,b)$ satisfy the following Bellman equations:

$$\begin{aligned} W_0(z_0,b) = \max_{\{c(t),h(t)\}} \int_0^\infty e^{-\rho t} \Big[& u[c(t), -h(t)] + \varepsilon \left[W_1(z(t),b) - W_0(z(t),b) \right] \\ & + \chi \left[\Omega_0(z(t),b) - W_0(z(t),b) \right] \Big] dt \end{aligned} \tag{3a}$$

$$\text{subject to } \dot{z}(t) = b + h(t) - c(t) - \pi z(t) + T, \;\; z(0) = z_0, \;\; \text{and } z(t), c(t), h(t) \geq 0$$

$$\begin{aligned} W_1(z_0,b) = \max_{\{c(t),h(t)\}} \int_0^\infty e^{-\rho t} \Big[& u[c(t), -h(t)] + \alpha \left[V(z(t),b) - W_1(z(t),b) \right] \\ & + \chi \left[\Omega_1(z(t),b) - W_1(z(t),b) \right] \Big] dt \end{aligned} \tag{3b}$$

$$\text{subject to } \dot{z}(t) = b + h(t) - c(t) - \pi z(t) + T, \;\; z(0) = z_0, \;\; \text{and } z(t), c(t), h(t) \geq 0$$

where the trajectories c_t and h_t are piecewise continuous and the trajectory $z(t)$ is continuous and

[10] These definitions guess ahead that state-0 households will want to buy bonds and state-1 households will want to sell, but they are general, as s_0 and s_1 could be negative and are only constrained by the assumption that private agents cannot short sell (effectively: create) either money or bonds in this economy.

piecewise differentiable.[11] The increase in real balances $\dot{z}$ is equal to the dividend income b plus the labor income $h(t)$ minus the expenditure flow $c(t)$ and the depreciation of real balances due to inflation $\pi z(t)$, plus finally the lump-sum transfer flow from the government.

Let $\mu_i(t)$ denote the costate variable associated with real balances, for households in state $i \in \{0,1\}$. Similarly, let $\beta_i(t)$ denote the costate variable associated with bonds. Given a path of expectations $\pi(t)$ and $q(t)$, the costates must satisfy the following Euler equations:

$$\rho\mu_0(t) = \dot{\mu}_0(t) - \pi(t)\mu_0(t) + \varepsilon[\mu_1(t) - \mu_0(t)] + \chi\left[\frac{\beta_0(t)}{q(t)} - \mu_0(t)\right] \tag{4a}$$

$$\rho\mu_1(t) = \dot{\mu}_1(t) - \pi(t)\mu_1(t) + \alpha[1 - \mu_1(t)] \tag{4b}$$

$$\rho\beta_0(t) = \dot{\beta}_0(t) + \mu_0(t) + \varepsilon[\beta_1(t) - \beta_0(t)] \tag{4c}$$

$$\rho\beta_1(t) = \dot{\beta}_1(t) + \mu_1(t) + \alpha[\beta_0(t) - \beta_1(t)] + \chi\left[q(t)\mu_1(t) - \beta_1(t)\right] \tag{4d}$$

These equations have straightforward interpretations. For example, the marginal flow value of real balances to households in state zero ($\rho\mu_0(t)$) can be decomposed as follows: first, real balances may gain value autonomously ($\dot{\mu}_0(t)$); second, they lose value to inflation ($-\pi(t)\mu_0(t)$); third, they gain value in transition to state 1 ($\varepsilon[\mu_1(t) - \mu_0(t)]$); and finally, they can be used to buy bonds at price $q(t)$ if the household is matched with a broker ($\chi\left[\beta_0(t)/q(t) - \mu_0(t)\right]$). The other equations admit analogous interpretations. The term $\mu_i(t)$ in the value of bonds represents the fact that these bonds pay a flow dividend of one unit of real balances per unit of time.

Equations (4) are necessary and sufficient for a solution to the household's problem together with the following transversality conditions:

$$\lim_{t\to\infty} e^{-(\rho+\pi(t)+\alpha)t}\mu_0(t)z(t) = 0 \tag{5a}$$

$$\lim_{t\to\infty} e^{-(\rho+\pi(t)+\alpha)t}\mu_1(t)z(t) = 0 \tag{5b}$$

$$\lim_{t\to\infty} e^{-rt}\beta_0(t) = 0 \tag{5c}$$

$$\lim_{t\to\infty} e^{-rt}\beta_1(t) = 0 \tag{5d}$$

If $\pi(t)$ and $q(t)$ are expected to converge to (π^s, q^s), then the only non-negative solution of the system (4) which satisfies (5) is convergence to the steady state $(\mu_0^s, \mu_1^s, \beta_0^s, \beta_1^s)$, which is defined to be the solution of (4) with $\pi(t) \equiv \pi^s$, $q(t) \equiv q^s$, and the time derivatives equal to zero.

Given the value of real balances to a household in state $i \in \{0,1\}$, fruit consumption and labor supply satisfy:

[11] I suppress additional arguments of the value functions to improve readability; the reader should bear in mind that the value functions depend on expectations over the full time paths of all exogenous and equilibrium variables.

$$u_1(c_i(t), -h_i(t)) = \mu_i(t) \quad \text{and} \quad u_2(c_i(t), -h_i(t)) = \mu_i(t) \tag{6}$$

As u is strictly concave in each argument, households with a high value of money work harder, consume less fruit, and therefore accumulate real balances faster than those with a low value of money. We can now prove a key property of the value functions:

Lemma 1. *Assume that $-T(t) < h_0(t) - c_0(t)$ and $-T(t) < h_1(t) - c_1(t)$ (so that all households can pay taxes out of pocket) and $\pi(t) > -\rho$ (so that $\mu_1 < 1$ and money is always spent given the opportunity) for all $t \geq 0$. Then the value functions W_0 and W_1 are linear in both arguments.*

Proof. See Appendix A. □

The fact that μ_i does not depend on the asset holdings of a household has two important consequences. By solving Equations (6), we can find (h_i, c_i), the choices of labor effort and fruit consumption of any household in state $i \in \{0,1\}$, which just like the value of money and bonds do not depend on the household's asset holdings. Furthermore, we can characterize the spending decisions of households matched in decentralized meetings as following a simple rule: depending on the price, and unless they are exactly indifferent, households either spend everything or nothing.

Lemma 2. *In matches with firms, households in state 1 buy the following amount of goods:*

$$d(z,b) = \begin{cases} 0 & \text{if } \mu_1 > 1 \\ \in [0,z] & \text{if } \mu_1 = 1 \\ z & \text{if } \mu_1 < 1 \end{cases} \tag{7}$$

In matches with brokers, households in state 0 buy the following amount of bonds:

$$s_0(z,b) = \begin{cases} z/q & \text{if } q < \beta_0/\mu_0 \\ \in [-b, z/q] & \text{if } q = \beta_0/\mu_0 \\ -b & \text{if } q > \beta_0/\mu_0 \end{cases} \tag{8}$$

In matches with brokers, households in state 1 sell the following amount of bonds:

$$s_1(z,b) = \begin{cases} -z/q & \text{if } q < \beta_1/\mu_1 \\ \in [-z/q, b] & \text{if } q = \beta_1/\mu_1 \\ b & \text{if } q > \beta_1/\mu_1 \end{cases} \tag{9}$$

Proof. See Appendix A. □

The following proposition verifies the guess that households in state 0 will buy bonds and households in state 1 will sell them in meetings with brokers, and households in state 1 will spend all of their money in meetings with firms.

Proposition 1. *In steady state, assuming $\pi > -\rho$ and $q \geq \beta_1/\mu_1$, the following inequalities hold:*

$$\mu_0 < \mu_1 < 1, \quad \frac{\beta_0}{\mu_0} > \frac{\beta_1}{\mu_1}, \quad \textit{and} \quad \frac{\beta_0}{\mu_0} > \frac{1}{\rho}.$$

In the special case of $q = \beta_1/\mu_1$, we additionally have: $\beta_1/\mu_1 < 1/\rho$.

Proof. See Appendix A. □

As one would expect, households in state 0 value real balances less than households in state 1. And *relative to real balances*, they value bonds more, so the direction of trade in the decentralized asset market is as expected. Furthermore, the reservation price of bonds for households in state 0 is always greater than the "fundamental price" $1/\rho$. We can say that this reservation price exhibits a "liquidity premium" because it helps such households store their wealth for future use in a way that avoids the inflation tax. In contrast, if the market price equals the reservation price of bonds for households in state 1, then it is smaller than $1/\rho$. We can interpret this as an "illiquidity discount" because such households would like to liquidate their bond holdings before the consumption opportunity arrives, but may not be able to do so.

2.4 Equilibrium

Let n_i denote the measure of households in state i. We must have $n_0 = 1 - n_1$, and transitions between states determine the following dynamic equation for n_1:

$$\dot{n}_1 = \varepsilon(1 - n_1) - \alpha n_1 \tag{10}$$

In equilibrium, the labor market, goods market, money market, and inter-dealer bond market must clear, and the government choices must satisfy its budget constraint. As the labor market is competitive, it clears if and only if the real wage is 1; this has already been incorporated into the household's problem. In order to describe aggregate flows through the other markets, let Z_i and B_i denote the *total* stocks of money and bonds held by households in state i.[12]

The goods and money markets clear if the flow of real balances from households to firms matches the flow of real balances in return (because of Walras' law, only this one equation is

[12] These totals are different from averages; for example, as the overall supply of bonds is B, we have $B_0 + B_1 = B$. We would have to write $n_0 B_0 + n_1 B_1 = B$ if B_0 and B_1 were averages.

necessary). Households in state 1 are matched with firms at flow rate α, and each such household spends all of its real balances, so the flow from households to firms is αZ_1. In return, households obtain real wage income at flow rate h_i, and spend some of it directly on goods at flow rate c_i, so the total flow of real balances from firms to households is $n_0(h_0 - c_0) + n_1(h_1 - c_1)$. The equality of these flows represents the demand for real balances and determines the value of money:

$$\alpha Z_1 = n_0(h_0 - c_0) + n_1(h_1 - c_1) \tag{11}$$

The unconstrained flow of real balances into the inter-dealer bond market is χZ_0, and the unconstrained inflow of bonds is χB_1. So if the candidate price $q = Z_0/B_1$ is one buyers are willing to pay and sellers are willing to receive, that is, $Z_0/B_1 \in [\beta_1/\mu_1, \beta_0/\mu_0]$, then this price clears the inter-dealer market. However, if the ratio of unconstrained flows is outside of this interval, then not every household can be served even if matched with a broker. Therefore, denote the probability that a household in state i gets served by $\psi_i \in [0,1]$.[13] Naturally, $\psi_i < 1$ can only be part of an equilibrium if $q = \beta_i/\mu_i$, that is, households on the long side of the market are indifferent between being served or not.[14] Bond market clearing can then be expressed as equality of the constrained flows of real balances and bonds:

$$\underbrace{\chi \psi_0 Z_0}_{\text{inflow of real balances}} = \underbrace{\chi \psi_1 B_1 q}_{\text{outflow of real balances}}$$

with solution:

$$q = \begin{cases} \beta_1/\mu_1 & \text{if } Z_0/B_1 < \beta_1/\mu_1 \\ Z_0/B_1 & \text{if } Z_0/B_1 \in [\beta_1/\mu_1, \beta_0/\mu_0] \\ \beta_0/\mu_0 & \text{otherwise} \end{cases} \tag{12a}$$

$$\psi_0 = \begin{cases} \frac{\beta_0/\mu_0}{Z_0/B_1} & \text{if } Z_0/B_1 > \beta_0/\mu_0 \\ 1 & \text{otherwise} \end{cases} \tag{12b}$$

$$\psi_1 = \begin{cases} \frac{Z_0/B_1}{\beta_1/\mu_1} & \text{if } Z_0/B_1 < \beta_1/\mu_1 \\ 1 & \text{otherwise} \end{cases} \tag{12c}$$

[13] Instead of a lottery where households get served with a probability, brokers could also offer households on the long side a rationed amount. This modeling choice only affects the distribution of assets across households *within* a state, but not the distribution of assets *between* states, which is all that matters for aggregate variables.

[14] Indifference also means that ψ_0 and ψ_1 do not enter the household's problem, because households expect a surplus from asset trade if and only if they expect to get served with probability 1.

The government must finance a flow of transfers T (or has access to taxes if $T < 0$) and dividend payments on the outstanding debt. As each unit of bonds pays a flow dividend of one unit of the numéraire good, equivalent to real balances, the total dividend flow is B. If the money supply grows at rate $\dot{M} = \gamma M$, then the government also has access to seigniorage revenue $\phi\dot{M}$, the real value of newly printed money. Using the definition of real balances, $Z \equiv \phi M$, we can express the seigniorage revenue as $\phi\dot{M} = Z\dot{M}/M = \gamma(Z_0 + Z_1)$. The budget constraint becomes:

$$T + B = \gamma(Z_0 + Z_1) \tag{13}$$

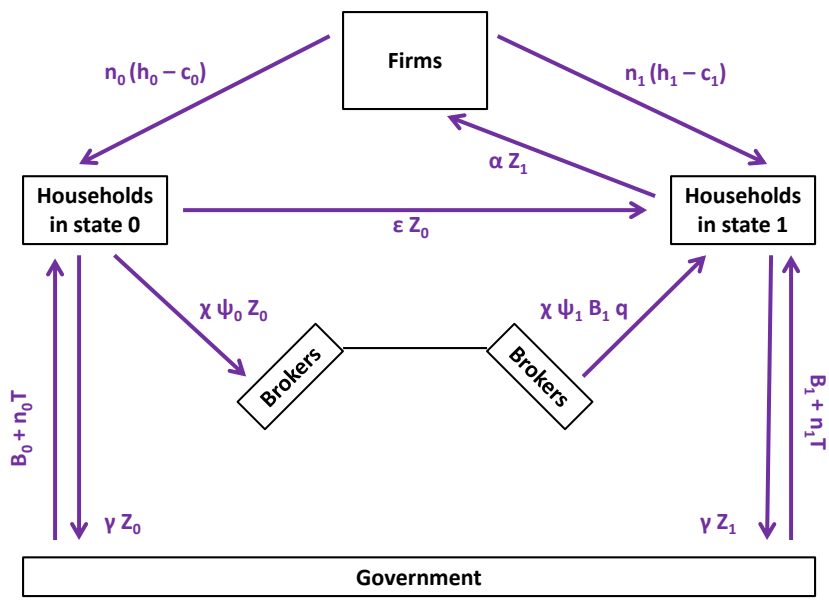


Figure 3: Flows of real balances between groups of agents

Figure 3 illustrates the flows of real balances between agents in the model. None of the firms, brokers, or government hold an inventory of assets, so equalizing inflows and outflows for these groups determines Equations (11), (12), and (13). What is left is to describe accumulation of assets by households. Fortunately, as explained above, all households in state $i \in \{0,1\}$ choose identical values of fruit consumption and labor effort, which we denote by the equilibrium per-household variables c_i and h_i. Accounting for the flow of assets to and from households in state 0 or state 1 is then straightforward:

$$\dot{Z}_0 = -\pi Z_0 + (B - B_1) + n_0(h_0 - c_0) + n_0 T - \varepsilon Z_0 - \chi\psi_0 Z_0 \tag{14a}$$

$$\dot{Z}_1 = -\pi Z_1 + B_1 + n_1(h_1 - c_1) + n_1 T + \varepsilon Z_0 - \alpha Z_1 + \chi\psi_0 Z_0 \tag{14b}$$

$$\dot{B}_1 = \varepsilon(B - B_1) - (\alpha + \chi\psi_1)B_1 \tag{14c}$$

For example, the stock of real balances held by households in state 0 increases due to dividend payments ($B_0 \equiv B - B_1$), labor effort ($n_0 h_0$), and the share of transfers going to these households ($n_0 T$); it decreases due to inflation (πZ_0), fruit consumption expenditure ($n_0 c_0$), transition to state 1 by some households who of course keep their money (εZ_0), and expenditure on bonds in matches with brokers ($\chi \psi_0 Z_0$). A flow equation for B_0 is redundant because $B_0 + B_1 = B$.

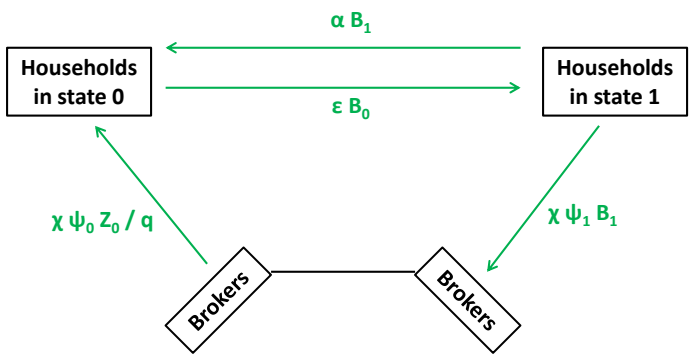


Figure 4: Flows of bonds between groups of agents

Definition 1. A strongly-monetary steady-state equilibrium is a vector $\{c_0, c_1, h_0, h_1, \mu_0, \mu_1, \beta_0, \beta_1, q, \psi_0, \psi_1, n_1, Z_0, Z_1, B_1, T\}$ which satisfies Equations (6) (consumption and labor effort are chosen optimally), (4) (the value functions represent the optimal values of bonds and money), (11) (the goods market clears), (12) (the bond market clears), $\phi = (Z_0 + Z_1)/M$ (the money market clears, and M is the supply of money), (13) (the government budget is in balance), and (14) (aggregate consistency), with all the time derivatives equal to zero, and in which $h_0 - c_0 + T > 0$.

In the literature, a "monetary" equilibrium is one in which all households *value* money ($\phi > 0$); here, I use the term "strongly-monetary" because the condition $h_0 - c_0 + T > 0$ requires that all households *accumulate* money. (It encompasses $h_1 - c_1 + T > 0$ because $\mu_1 > \mu_0$.) The reason is technical. If some households did not want to *accumulate* money, they might still *value* it for relaxing their budget constraint. But it can be shown that such households would decumulate money to hit the constraint $z \geq 0$ in finite time; as a result, their value function will not be linear in money, their willingness to pay for assets will be heterogeneous, and their decisions cannot be aggregated in the simple form shown above. We would need a much richer variable space merely to define an equilibrium.[15]

Also note that in the definition above, government transfers T are treated as an endogenous variable that must satisfy the government budget constraint, while government debt B and the

[15] There is no simple condition on exogenous parameters equivalent to $h_0 - c_0 + T > 0$, but there are some sufficient conditions. See Appendix A.

rate of money growth γ are exogenous. We could define an analogous equilibrium where T is exogenous and either B or γ are endogenous to satisfy Equation (13).

In order to define and characterize dynamic equilibria, we need to describe inflation expectations $\pi(t) = -\dot{\phi}(t)/\phi(t)$. Because $\phi \equiv Z/M$ (the real price of money equals real balances divided by the money supply), and M grows at rate γ, we can derive:

$$\pi = \gamma - \frac{\dot{Z}_0 + \dot{Z}_1}{Z_0 + Z_1}$$

We cannot just use Equations (14) to replace both $\dot{Z}_0$ and $\dot{Z}_1$, because doing so would just yield the goods market clearing condition. We need to find an independent equation, and the solution is to differentiate the goods market clearing condition with respect to time:

$$\alpha \dot{Z}_1 = -\dot{n}_1(h_0 - c_0) + \dot{n}_1(h_1 - c_1) + (1 - n_1)(\dot{h}_0 - \dot{c}_0) + n_1(\dot{h}_1 - \dot{c}_1) \tag{15}$$

Next, we differentiate with respect to time the system (6), for $i = 0, 1$:

$$\dot{c}_i = \frac{u_{22}(c_i, -h_i) - u_{21}(c_i, -h_i)}{|H_u(c_i, -h_i)|}\dot{\mu}_i \qquad \text{and} \qquad \dot{h}_i = \frac{u_{12}(c_i, -h_i) - u_{11}(c_i, -h_i)}{|H_u(c_i, -h_i)|}\dot{\mu}_i \tag{16}$$

where $|H_u(c_i, -h_i)|$ denotes the determinant of the Hessian matrix of u, evaluated at $(c_i, -h_i)$. Finally, we can use Equations (4) to substitute for $\dot{\mu}_i$ and Equation (10) for $\dot{n}_1$, and write:

$$\pi = \gamma - \underbrace{\left[\frac{\dot{Z}_0 + \dot{Z}_1}{Z_0 + Z_1}\right]}_{\substack{\text{all time derivatives substituted using (14a) for } \dot{Z}_0 \\ \text{and (4a,b), (16), (15), and (10) for } \dot{Z}_1}} \tag{17}$$

With the hard work done, we can now describe a dynamic equilibrium purely in terms of ordinary differential equations, contemporaneous equations, and transversality conditions.

Definition 2. A strongly-monetary dynamic equilibrium is a vector of paths $\{c_0(t), c_1(t), h_0(t), h_1(t), \mu_0(t), \mu_1(t), \beta_0(t), \beta_1(t), q(t), \psi_0(t), \psi_1(t), n_1(t), Z_0(t), Z_1(t), B_1(t), T(t)\}$ which satisfy equations (4), (5), (6), (11), (12), (13), (14), and (17), and $h_0(t) - c_0(t) + T(t) > 0$ for all $t \geq 0$. The exogenous variables may be paths as well, provided they are bounded, piecewise continuous, and common knowledge among all agents.

2.5 Analysis of equilibrium

As is standard in models of this kind, money is not superneutral. The inflation rate π equals the money growth rate γ in steady state, and through Equations (4), inflation affects the value of money, which in turn determines the choices of fruit consumption and labor effort.

The fact that households are heterogeneous with regard to their value of money makes the classical question of monetary neutrality interesting and non-trivial. First, money is neutral in the long run if the dynamic equilibrium is unique. But in the short run, money is generally not neutral. To see this, start in any equilibrium, and deliver newly printed money to some or all households. Unless money is delivered to state-0 and state-1 households in exact proportion to the previous totals, Z_0 and Z_1, the ratio Z_1/Z_0 must change. An increase in the price level proportional to the change in the money supply cannot restore the old level of Z_1, so the goods market clearing equation (11) will not be satisfied. Even an increase in the price level sufficent to exactly restore the old level of Z_1 cannot restore the old equilibrium. To see why, assume that the price level does adjust to keep Z_1 constant after the money injection in order to satisfy goods market clearing. If the new level of Z_0 is below (above) the old level, households will expect it to increase (decrease), implying temporary deflation (inflation). Consequently, the choices of fruit consumption and labor effort will change, and goods market clearing is not satisfied after all, a contradiction. To summarize: a money injection will (almost surely) affect both Z_0 and Z_1, cause expected inflation or deflation as Z_1 returns to steady state, and thereby affect the choices of households along the transition path.

We can carry the analysis further by considering a traditional "helicopter drop": a money injection equally delivered to all households. For simplicity, we assume that the economy was in a steady-state equilibrium before the injection. Such a helicopter drop will *compress the distribution of money holdings*, and while the distribution of money holdings among households in the same state has no effect due to the linear value functions, the distribution of money holdings between households in different states matters because households in state 1 have earlier opportunities to spend money. As a result, they hold more money (Z_1/n_1 on average) than households in state 0 (Z_0/n_0 on average) in the steady-state equilibrium. The helicopter drop compresses this distribution and reduces the ratio Z_1/Z_0. As a result, Z_1 falls compared with the steady state, and Z_0 rises; as a consequence, expenditure on the lumpy consumption good falls, and accumulation of money falls as households work less and use more of their income on immediate consumption of the numéraire good. Crucially, however, households in state 0 seek to convert their temporary windfall of real balances (high Z_0) into assets that offer a better store of value: in the baseline model, the only option are government bonds. So at least in the interior region of asset market equilibrium where $q = Z_0/B_1$, the market price of government bonds will increase. An econometrician will observe this as a helicopter drop of money causing a temporary fall in real interest rates. This fall in interest rates may stimulate investment and output, as I show in Section 3. But contrary to tradi-

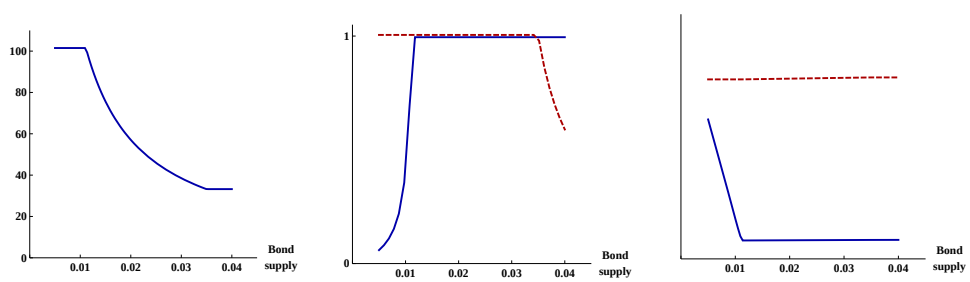
tional Baumol-Tobin intuition, the helicopter drop also has a *negative direct effect* on consumption demand, output, and welfare, as real balances are less efficiently distributed.

2.6 Comparative statics with respect to the bond supply

The comparative statics of steady-state equilibrium with respect to the bond supply B are important because they help understand the twin roles these bonds play: they are better saving vehicles than money, but they also provide indirect liquidity services because households can liquidate them when they expect to need money soon. To begin with, I assume that the money growth rate γ is fixed, and that the flow of lump-sum transfers, T, adjusts to satisfy the government budget constraint. A look at the asset market clearing equations (12) suggests that there are three regions to consider, and the total bond holdings by households in state 1, B_1, is a crucial variable in determining which region equilibrium falls into. The flow equation (14c) reveals that in steady state, B_1 is a constant proportion of the bond supply B:

$$B_1 = \frac{\varepsilon}{\varepsilon + \alpha + \chi \psi_1} B.$$

As $\psi_1 \in (0,1]$, we can see that households in state 0 hold more bonds on average than those in state 1 because the steady-state measure of households in state 1 is $\varepsilon/(\varepsilon+\alpha)$.



Market price of bonds (fundamental value = 33.3)

Matching probabilities for households in state 0 and state 1 (dashed)

Real balance totals for households in state 0 and state 1 (dashed)

Figure 5: Comparative statics of the bond supply, under the assumption that the inflation rate π is fixed and the flow of transfers T is endogenous. Key parameters: $\rho = .03$, $\gamma = \pi = 0$, $\varepsilon = .5$, $\alpha = 1$, $\chi = 6$.

Figure 5 shows the effect of bond supply on some important equilibrium objects. The first region of the asset market equations is where the supply of bonds by households in state 1 is too large for the demand by households in state 0. Equilibrium is in this region if B is large, and in this case $q = \beta_1/\mu_1$ (the market price equals the reservation price of bond sellers) and $\psi_1 < 1$ (bond sellers are rationed). In this region, small changes in B have no effect on the equilibrium.

The second region of the asset market equations is where the supply of bonds is interior, so that $q = Z_0/B_1$. In this case, $\psi_0 = \psi_1 = 1$ (all asset market participants are served) and $B_1 = \varepsilon/(\varepsilon+\alpha+\chi)B$, so an increase in bond supply directly decreases q. Using the Euler equations (4), we can establish that this decrease in q causes μ_0 to rise while μ_1 is unaffected; converting money into bonds becomes cheaper for households in state 0, and they are therefore willing to work harder, consume less of the fruit consumption good, and accumulate more money. By the goods market clearing equation (11), the extra production causes Z_1 to increase, and if the money supply has not changed, this is achieved through a fall in the price level. The end result of an increase in bond supply in this region is lower prices, lower consumption of the numéraire good but higher consumption of the lumpy good, higher output, and higher welfare.[16]

The third region of the asset market equations is where the supply of bonds by households in state 1 is so small that the demand by households in state 0 cannot be satisfied. Equilibrium is in this region if B is small, and in this case $q = \beta_0/\mu_0$ (the market price equals the reservation price of bond buyers) and $\psi_0 < 1$ (bond buyers are rationed). Small changes in B have no effect on prices, consumption, production, or welfare, just like in the first region when the bond supply was large. In comparison to the first region, output and welfare are lower if the bond supply is small. The intuition is that these bonds provide a useful service: they help households in state 0 store their wealth in such a way that avoids the inflation tax. As a result, such households are willing to accumulate wealth faster. Limiting the bond supply drives down yields, and may encourage households to invest in alternative assets such as physical capital (see Section 3), but through the channel of the aggregate supply of liquidity a lower bond supply reduces output.

However, it is worth noting that all of the previous analysis makes the assumption that the government is committed to a certain growth rate of the money supply, and adjusts its tax/transfer balance to satisfy the government budget constraint. While common in monetary theory, this assumption is not quite realistic. An alternative would be to assume that the government is committed to a certain flow of taxes and transfers, possibly including debt service, so that either the budget deficit $T+B$ or the structural deficit T are held constant even as the total stock of debt, B, changes.

In a monetary model with a representative household, there is not much difference between these two assumptions. But here, there is a big difference, because the *distribution* of real balances affects the *level* of real balances households end up holding in equilibrium. For example, in the region where the supply of bonds is so low that bond buyers are rationed ($B \to 0$ and therefore $\psi_0 < 1$), the level of real balances held by households in state 0 is very responsive to changes in the supply of bonds; the third panel of Figure 5 provides the illustration. The reason is that households in state 0 are willing to accumulate money, but they would prefer to hold their wealth

[16] Welfare is higher because with the increase in μ_0, equilibrium moves closer to the first-best. The first-best would be attained if μ_0 and μ_1 were equal and maximal at 1, the marginal utility of consumption of the lumpy good.

in bonds which have a better rate of return. However, if the supply of bonds is small, they may not be able to obtain as many bonds, or not as quickly, as they would like. Therefore, they will hold a higher proportion of the total money supply than if the supply of bonds were larger. The next step of the argument is crucial: it is not the total money supply that determines the price level via goods market clearing, but the *quantity of money held by households looking to spend money* on goods (Equation 11). Household heterogeneity is clearly essential for this point. If the households about to spend money hold less of it, then the aggregate price level is lower, the total of real balances in the economy is higher, and the velocity of circulation is lower.

If the government happens to be committed to transferring real balances to households (net of taxes) at a fixed flow rate, rather than as a fixed proportion of overall real balances, then the inflation rate is endogenously determined by the ratio of the monetary government deficit to the amount of money households are willing to hold:

$$\gamma = \frac{T+B}{Z_0+Z_1}$$

Consequently, in the region where the bond supply is so low that bond buyers are rationed, a small increase in the bond supply will reduce the level of real balances households are willing to hold, and will increase the inflation rate.[17] Even if the government keeps the budget deficit $T+B$ constant (it raises lump-sum taxes to finance the additional debt service), so that the required seigniorage revenue $\gamma(Z_0+Z_1)$ remains constant, the decrease in total real balances implicitly raises the inflation rate. As the price of bonds is increasing in inflation in the region where buyers are rationed (Appendix B), we are left with the counterintuitive result that the demand curve for bonds is upward-sloping when the bond supply is low and the inflation rate is endogenous.

Figure 6 illustrates this conclusion for two cases: first, when the government keeps the real budget deficit $T+B$ fixed at a positive number; second, when the government keeps the real *structural deficit* T, defined as expenditures minus revenues excepting payments for debt service, fixed. In the latter case, additional debt service must be financed by seigniorage revenue instead of lump-sum taxes, so naturally, inflation responds to the supply of bonds directly, not just through Z_0.

The comparative statics of the baseline model with respect to variations in inflation and bond liquidity are illustrative but tangential, and are therefore relegated to Appendix B.

[17] Strictly speaking, this is only true if $T+B>0$, i.e. the government is monetizing a deficit. The argument is reversed when the government is running a surplus and seigniorage is negative.

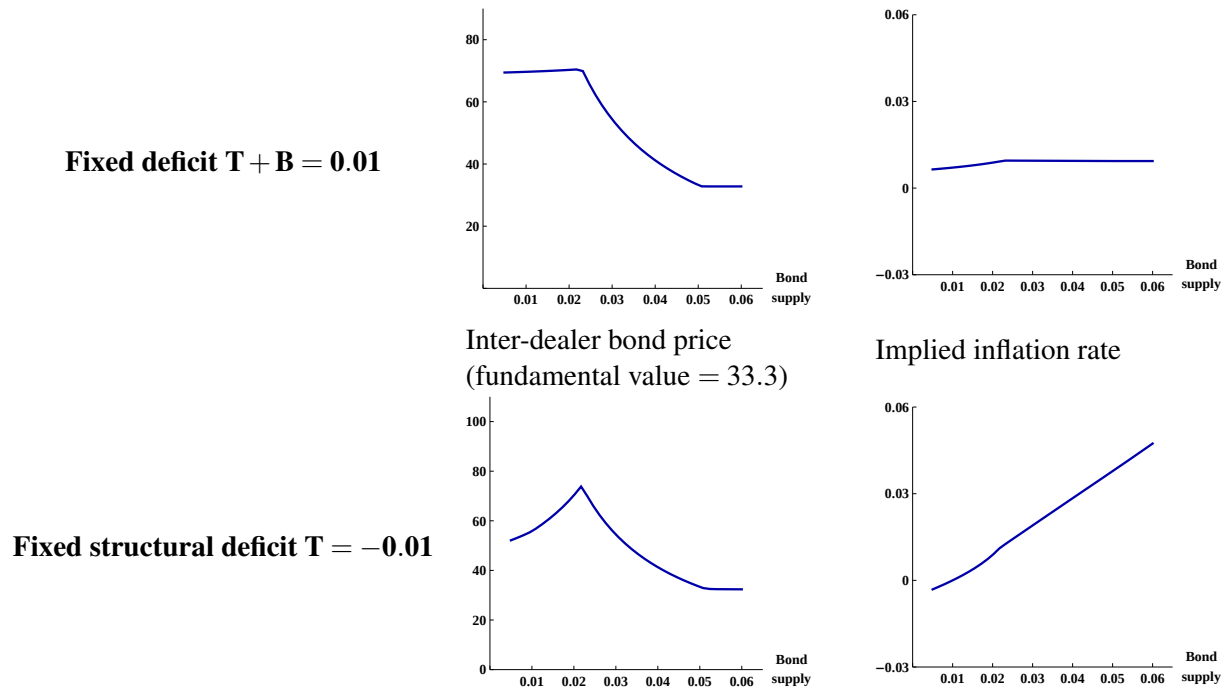


Figure 6: Comparative statics of the bond supply, under the assumption that the inflation rate π is endogenous. Key parameters: $\rho = .03$, $\varepsilon = .5$, $\alpha = 1$, $\chi = 1$.

3 A Theory of Quantitative Easing

So far, I have presented a minimal version of the model in order to describe its engine in a clean and transparent way, and to elucidate the central results. The main value of the framework, however, derives from its tractability and flexibility; it can accommodate a variety of fruitful extensions. The goal of this section is to extend the model in such a way that it can speak to the effects of the recent quantitative easing policies pursued in the U.S. and proposed in other countries.[18] To this end, I will introduce two extensions that are key to understanding quantitative easing: first, how intervention in the market for government bonds can affect the yield on these bonds, and second, how this will in turn affect investment and output.

3.1 Government intervention in asset markets

Begin with the model as described so far and its two assets, money and bonds. The government can create and sell new bonds at flow rate $S \in \mathbb{R}$ in the decentralized bond market; if $S > 0$, the

[18] For the purpose of this paper, "quantitative easing" is defined as a program intended to reduce the yields of illiquid assets which satisfies two criteria. First, the central bank directly purchases these illiquid assets or what it believes to be close substitutes. Second, the central bank targets quantities, not prices or yields.

intervention is an open-market sale, and if $S < 0$, an open-market purchase.[19] As long as the flow of neither open-market sales nor purchases is too large, specifically $S \in [-\chi B_1, \chi Z_0/q]$, then the intervention changes the magnitude of private asset flows but not their direction. In what follows, I will make the assumption that this is the case, although it would in principle be possible for the government to purchase bonds at such a pace (and, correspondingly, such an attractive price) that even households in state 0 would want to sell them for money.

The unconstrained flow of private demand for bonds is still $\chi Z_0/q$, and the unconstrained flow of private supply of bonds is still χB_1. The government adds a flow supply of S (a flow demand if negative). It is still possible that households on the long side of the market are rationed. As in the simplified model, bond market clearing can be expressed as equality of the constrained flows of real balances and bonds:

$$\chi \psi_0 Z_0/q = \chi \psi_1 B_1 + S$$

The solution is also similar:

$$q = \begin{cases} \beta_1/\mu_1 & \text{if } \chi Z_0/(\chi B_1 + S) < \beta_1/\mu_1 \\ \chi Z_0/(\chi B_1 + S) & \text{if } \chi Z_0/(\chi B_1 + S) \in [\beta_1/\mu_1, \beta_0/\mu_0] \\ \beta_0/\mu_0 & \text{otherwise} \end{cases} \tag{18a}$$

$$\psi_0 = \begin{cases} \frac{\beta_0}{\mu_0} \frac{\chi B_1 + S}{\chi Z_0} & \text{if } \chi Z_0/(\chi B_1 + S) > \beta_0/\mu_0 \\ 1 & \text{otherwise} \end{cases} \tag{18b}$$

$$\psi_1 = \begin{cases} \frac{\chi Z_0 \frac{\mu_1}{\beta_1} - S}{\chi B_1} & \text{if } \chi Z_0/(\chi B_1 + S) < \beta_1/\mu_1 \\ 1 & \text{otherwise} \end{cases} \tag{18c}$$

The government budget must also account for these new flows, in addition to the flow of transfers and dividend payments on the bonds held by households. On the income side, the government has access to seigniorage revenue and to the receipts from open-market sales of bonds, qS. The budget constraint becomes:

$$T + B_0 + B_1 = \gamma(Z_0 + Z_1) + qS \tag{19}$$

[19] Alternatively, newly created bonds could be sold in the integrated competitive market for labor, money, and the numéraire good; this would correspond to the bonds being auctioned at first before trading on secondary markets thereafter. For many assets, this is realistic: see Geromichalos and Herrenbrueck (2016) for an in-depth discussion. In the present model, households in state 0 value bonds more relative to money than households in state 1; therefore, as long as the rate of issue was not too large, only state-0 households would buy the newly issued bonds, just as they do when bonds (of any age) are sold in the decentralized bond market. However, in the competitive market, the issue price would be equal to the marginal rate of substitution for state-0 households, β_0/μ_0, which is never less than the bond market price q.

The accumulation of assets by households is subject to minor changes as a consequence. Again, all households in state $i \in \{0,1\}$ choose identical values of fruit consumption and labor effort, which we denote by the equilibrium per-household variables c_i and h_i.

$$\dot{Z}_0 = -\pi Z_0 + B_0 + n_0(h_0 - c_0) + n_0 T - \varepsilon Z_0 - \chi \psi_0 Z_0 \tag{20a}$$

$$\dot{Z}_1 = -\pi Z_1 + B_1 + n_1(h_1 - c_1) + n_1 T + \varepsilon Z_0 - \alpha Z_1 + \chi \psi_0 Z_0 - qS \tag{20b}$$

$$\dot{B}_0 = -\varepsilon B_0 + (\alpha + \chi \psi_1) B_1 + S \tag{20c}$$

$$\dot{B}_1 = \varepsilon B_0 - (\alpha + \chi \psi_1) B_1 \tag{20d}$$

The definition of steady-state and dynamic equilibria is analogous to Section 2.4. A steady state exists only if $S = 0$, as the total supply of bonds would have to be constant over time.

Let us next analyze a temporary intervention, where $S < 0$ for a finite period of time (an open-market purchase of illiquid bonds). For simplicity, assume that the economy was previously in a steady-state equilibrium with no expectations of intervention, and that the asset market was in the interior region where $q = Z_0/B_1$. Then, the government intervenes in the inter-dealer asset market to buy bonds for money, driving up bond prices to $q = \chi Z_0/(\chi B_1 + S)$. During the intervention, and relative to the old steady state, Z_1 is higher, Z_0 and B_0 are lower, and B_1 is unaffected (to a first approximation). The rise in Z_1 implies more consumption of the lumpy good and higher output during the intervention, but also a rise in the price level to absorb some of the extra demand. It is this rise in the price level that causes Z_0 to fall.

Once the intervention has concluded, however, and $q = Z_0/B_1$ holds again, the temporary fall in Z_0 will *depress* bond prices along the transition path to the new steady state. Because the supply of bonds available to households is lower in the new steady state than in the initial steady state, bond prices q will ultimately be higher. There are consequently two counteracting forces on q after the intervention: the need to converge to a higher level in the new steady state, and the lack of demand for bonds due to a temporary depression of Z_0. Which one prevails is a quantitative question. In summary, an open-market purchase of illiquid assets causes lower yields both during the intervention and in the long run, although possibly not along the entire transition path. In addition, the purchase has a positive direct effect on output in the short run because real balances are more efficiently distributed, but a negative direct effect on output in the long run because scarce bonds perform a useful service in this economy.

It is instructive to contrast the effects of an open-market purchase with those of a helicopter drop, where the fiscal authority sends transfers to all households and the central bank monetizes the cost. Recall that a helicopter drop compresses the distribution of real balance holdings and redistributes real balances from those more likely to spend it to those less likely. An open market purchase, by contrast, directs money better because households reveal their valuations through

their decision to either buy or sell assets. The direct effects on output therefore go in opposite directions. However, they are both capable of reducing the yields on government bonds, and as the next section shows, stimulate capital accumulation and output indirectly.

3.2 Capital accumulation

In the previous sections, we have seen several ways in which monetary policy can exert direct influence on the real economy, through temporary portfolio effects or the supply of saving vehicles. However, most economists (and their textbooks) treat these channels as secondary, and instead emphasize how monetary policy affects the economy *through its effects on interest rates*. The well-known wrinkle is that yields on government bonds do not represent the cost of funds for firms looking to expand production. By introducing physical capital as a partially liquid asset the model can address this question directly.

In the full model, there are now three durable commodities: money, real bonds, and physical capital. In contrast to the other two assets, physical capital depreciates at rate $\delta > 0$. New physical capital is accumulated by households willing to invest units of the numéraire good. In steady-state equilibrium, households in both states will invest in new capital, but because households in state 1 value capital less relative to money, they will invest less than necessary to cover depreciation, while households in state 0 will invest more, and the aggregate stock of capital remains constant. Capital accumulation is subject to a friction: if $i(t)$ denotes the flow of resources invested by a household, and $k(t)$ denotes the stock of capital held by the same household, then new capital is created at a flow rate $\Phi[i(t)/k(t)]k(t)$, where Φ is a strictly concave function.

$$\dot{k}(t) = -\delta k(t) + \Phi\left(\frac{i(t)}{k(t)}\right)k(t) \tag{21}$$

The reason to include this capital accumulation friction in the model is that without it, if the value of capital changed, households would want to jump discretely to the new equilibrium capital stock. In continuous time, that would imply infinite investment concentrated into an infinitesimal amount of time – technically not feasible. For the model, we will use the following function Φ:

$$\Phi\left(\frac{i}{k}\right) = \frac{\delta^{\nu}}{1-\nu}\left(\frac{i}{k}\right)^{1-\nu} - \frac{\delta\nu}{1-\nu}, \tag{22}$$

where δ is the rate of depreciation, and $\nu \geq 0$ is a parameter that governs how difficult it is to change the rate of capital accumulation.[20] If $\nu = 0$, then there is no friction. The functional

[20] Expressing capital adjustment costs in this fashion goes back to Lucas and Prescott (1971) and is standard in the business cycle literature (e.g., Francis and Ramey, 2005).

form has the important feature that even though the efficacy of investment is strictly concave, the household's value function still turns out to be linear in all three assets.

Once accumulated, households rent their labor and capital to firms who use it subject to a standard production function with capital share $\xi \in (0,1)$ and total factor productivity $A > 0$:

$$f(k,h) = Ak^{\xi}h^{1-\xi} \tag{23}$$

Capital and labor services are traded in competitive spot markets, so the rental rate R and the real wage ω equal the marginal products of capital and labor.

However, just like bonds, capital is not perfectly liquid as an asset; once accumulated, households can only sell it in a decentralized equity market analogous to the trade in bonds. In order to keep notation transparent, re-denote by χ^B the arrival rate of matches between households and brokers in the decentralized bond market, and denote by χ^K the analogous arrival rate in the decentralized equity market.[21] Similarly, index the market outcome variables q, ψ_0, ψ_1 by superscripts B and K, respectively.

Because the real wage is no longer constant, we have to adapt Equations (6). Given the value of real balances to households in states 0 and 1, and the real wage ω, the representative choices of fruit consumption and labor supply satisfy:

$$u_1(c_0(t), -h_0(t)) = \mu_0(t) \qquad u_2(c_0(t), -h_0(t)) = \mu_0(t)\omega(t) \tag{24a}$$

$$u_1(c_1(t), -h_1(t)) = \mu_1(t) \qquad u_2(c_1(t), -h_1(t)) = \mu_1(t)\omega(t) \tag{24b}$$

Denote by $\kappa_i(t)$ the costate variable associated with physical capital for a household in state $i \in \{0,1\}$. Standard Tobin's-q algebra yields that the household's rate of investment i, the value of capital, and the value of real balances are related as follows:

$$\Phi'\left(\frac{i}{k}\right) = \frac{\mu_i}{\kappa_i}$$

$$\rho\kappa_i = \dot{\kappa}_i + R\mu_i + \left[-\delta + \Phi\left(\frac{i}{k}\right) - \frac{i}{k}\Phi'\left(\frac{i}{k}\right)\right]\kappa_i + [\text{transitions}]$$

It follows that households invest in proportion to their capital holdings, in such a way that their capital holdings increase net of depreciation if and only if the marginal value of capital exceeds that of real balances:

$$i = \left(\frac{\kappa_i}{\mu_i}\right)^{1/\nu}\delta k. \tag{25}$$

[21] This assumption is made in order to treat all assets symmetrically. One could take the limit $\chi^K \to \infty$. But physical capital is often made-for-purpose and subject to private information, which is one reason financial firms are in business converting illiquid capital into less illiquid claims to it. The model abstracts from these details.

The household-level variables i and k can be eliminated from the Euler equations for the value of capital. Consequently, value functions are linear in a household's capital holdings. Taking as given R, the rental rate on capital, and q^K, the secondary market price of capital, the value of capital to households in state 0 or 1 satisfies:

$$\rho\kappa_0 = \dot{\kappa}_0 + R\mu_0 - \frac{1-\nu\left[\kappa_0/\mu_0\right]^{(1-\nu)/\nu}}{1-\nu}\delta\kappa_0 + \varepsilon(\kappa_1 - \kappa_0) \tag{26a}$$

$$\rho\kappa_1 = \dot{\kappa}_1 + R\mu_1 - \frac{1-\nu\left[\kappa_1/\mu_1\right]^{(1-\nu)/\nu}}{1-\nu}\delta\kappa_1 + \alpha(\kappa_0 - \kappa_1) + \chi^K\left[q^K\mu_1 - \kappa_1\right] \tag{26b}$$

The Euler equation for money accumulation by households in state 0 changes, too, since these households can now use their money to buy both bonds and capital on decentralized markets. The other three of the original Euler equations do not change, but for completeness I include them here:

$$\rho\mu_0 = \dot{\mu}_0 - \pi\mu_0 + \varepsilon[\mu_1 - \mu_0] + \chi^B\left[\frac{\beta_0}{q^B} - \mu_0\right] + \chi^K\left[\frac{\kappa_0}{q^K} - \mu_0\right] \tag{27a}$$

$$\rho\mu_1 = \dot{\mu}_1 - \pi\mu_1 + \alpha[1 - \mu_1] \tag{27b}$$

$$\rho\beta_0 = \dot{\beta}_0 + \mu_0 + \varepsilon[\beta_1 - \beta_0] \tag{27c}$$

$$\rho\beta_1 = \dot{\beta}_1 + \mu_1 + \alpha[\beta_0 - \beta_1] + \chi^B\left[q^B\mu_1 - \beta_1\right] \tag{27d}$$

As before, in order to keep the model tractable, it needs to be the case that households in state 0 do accumulate some money. In other words, capital accumulation cannot be too rapid for any household, and for all numerical calculations this condition must be verified. A simple sufficient condition would be that the real balances earned as capital income exceed the real balances invested in new capital. Since households invest in proportion to how much capital they hold, we need to make the following assumption on equilibrium outcomes:

$$R \geq (\kappa_0/\mu_0)^{1/\nu}\,\delta \tag{28}$$

Again, there is no equivalent condition purely in terms of exogenous parameters. But the condition will be satisfied if $\mu_0 \to \mu_1$ (in other words, for low enough inflation, because then capital is not valued for its liquidity properties, and we have both $\kappa_0/\mu_0 \to 1$ and $R \to \rho + \delta$) or if $\nu \to \infty$ (households exactly invest to replace their depreciating capital, no more and no less).

3.3 Equilibrium with capital accumulation

The aggregation of household choices follows the same principles as in Section 2. Let us begin with the aggregate flows of capital between households in state 0 and state 1. Flows due to transition

between states and due to trade are standard by now, but we also have to account for accumulation. For example, a household in state 0 with capital holdings k loses a flow of δk to depreciation, but it spends i units of the numéraire good on investment. This household will therefore accumulate capital at the following rate:

$$\begin{aligned}\dot{k} &= -\delta k + \Phi\left(\frac{i}{k}\right)k \;=\; -\delta k + \Phi\left(\delta(\kappa_0/\mu_0)^{1/\nu}\right)k \\ &= \frac{\left(\frac{\kappa_0}{\mu_0}\right)^{(1-\nu)/\nu} - 1}{1-\nu}\delta k\end{aligned}$$

after some algebra, and a household in state 1 will make analogous choices. Consequently, if we use K_0 and K_1 to denote the total stocks of capital held by households in states 0 and 1, respectively, then K_0 and K_1 must satisfy:

$$\dot{K}_0 = \frac{\left(\frac{\kappa_0}{\mu_0}\right)^{(1-\nu)/\nu} - 1}{1-\nu}\delta K_0 - \varepsilon K_0 + \left(\alpha + \chi^K \psi_1^K\right) K_1 \tag{29a}$$

$$\dot{K}_1 = \frac{\left(\frac{\kappa_1}{\mu_1}\right)^{(1-\nu)/\nu} - 1}{1-\nu}\delta K_1 + \varepsilon K_0 - \left(\alpha + \chi^K \psi_1^K\right) K_1 \tag{29b}$$

Next, we need to describe the accumulation of money and bonds by households. As in the model of Section 2, all households in state $i \in \{0,1\}$ choose identical values of fruit consumption and labor effort, which we denote by the equilibrium per-household variables c_i and h_i. Four things change, however. First, the real wage is no longer equal to 1. Second, households now derive income from their capital holdings. Third, they also spend some of their income on investment goods, and finally, they participate in decentralized trade for existing units of capital.

$$\begin{aligned}\dot{Z}_0 = -\pi Z_0 + (B - B_1) + \left[R - \delta\left(\frac{\kappa_0}{\mu_0}\right)^{1/\nu}\right]K_0 \\ + n_0(\omega h_0 - c_0) + n_0 T - \varepsilon Z_0 - \left(\chi^B \psi_0^B + \chi^K \psi_0^K\right) Z_0\end{aligned} \tag{30a}$$

$$\begin{aligned}\dot{Z}_1 = -\pi Z_1 + B_1 + \left[R - \delta\left(\frac{\kappa_1}{\mu_1}\right)^{1/\nu}\right]K_1 \\ + n_1(\omega h_1 - c_1) + n_1 T + \varepsilon Z_0 - \alpha Z_1 + \left(\chi^B \psi_0^B + \chi^K \psi_0^K\right) Z_0\end{aligned} \tag{30b}$$

$$\dot{B}_1 = \varepsilon(B - B_1) - \left(\alpha + \chi^B \psi_1^B\right) B_1 \tag{30c}$$

The market clearing conditions for the inter-dealer bond market are the same as before, given by Equations 12 if there is no government intervention and by Equations 18 if there is, but now

with ψ_0, ψ_1, and q indexed by superscript B. The market clearing conditions for the inter-dealer market for existing capital (without intervention) are analogous:

$$q^K = \begin{cases} \kappa_1/\mu_1 & \text{if } Z_0/K_1 < \kappa_1/\mu_1 \\ Z_0/K_1 & \text{if } Z_0/K_1 \in [\kappa_1/\mu_1, \kappa_0/\mu_0] \\ \kappa_0/\mu_0 & \text{otherwise} \end{cases} \tag{31a}$$

$$\psi_0^K = \begin{cases} \frac{\kappa_0/\mu_0}{Z_0/K_1} & \text{if } Z_0/K_1 > \kappa_0/\mu_0 \\ 1 & \text{otherwise} \end{cases} \tag{31b}$$

$$\psi_1^K = \begin{cases} \frac{Z_0/K_1}{\kappa_1/\mu_1} & \text{if } Z_0/K_1 < \kappa_1/\mu_1 \\ 1 & \text{otherwise} \end{cases} \tag{31c}$$

Goods market clearing now accounts for the neoclassical production function and investment:

$$\alpha Z_1 + n_0 c_0 + n_1 c_1 + \left(\frac{\kappa_0}{\mu_0}\right)^{1/\nu} \delta K_0 + \left(\frac{\kappa_1}{\mu_1}\right)^{1/\nu} \delta K_1 = A\left(K_0 + K_1\right)^{\xi} \left(n_0 h_0 + n_1 h_1\right)^{1-\xi} \tag{32}$$

The spot markets for capital and labor services clear at competitive prices:[22]

$$R = A\xi \left(\frac{K_0 + K_1}{n_0 h_0 + n_1 h_1}\right)^{\xi - 1} \qquad \omega = A(1-\xi)\left(\frac{K_0 + K_1}{n_0 h_0 + n_1 h_1}\right)^{\xi}$$

The system of capital flows (29) is linear homogeneous in (K_0, K_1). Consequently, $K_0 = K_1 = 0$ is one steady-state solution to this system, but it would imply that there is no production, and hence no monetary economy. But because of homogeneity, all steady-state solutions satisfy that the determinant of the system is zero, and from this we can derive two steady-state equations:

$$\left[\left(\frac{\kappa_0}{\mu_0}\right)^{(1-\nu)/\nu} - 1\right]\left(\alpha + \chi^K \psi_1^K\right) = \left[1 - \left(\frac{\kappa_1}{\mu_1}\right)^{(1-\nu)/\nu}\right]\left[\varepsilon - \delta \frac{\left(\frac{\kappa_0}{\mu_0}\right)^{(1-\nu)/\nu} - 1}{1-\nu}\right] \tag{33a}$$

$$\left[\left(\frac{\kappa_0}{\mu_0}\right)^{(1-\nu)/\nu} - 1\right] K_0 = \left[1 - \left(\frac{\kappa_1}{\mu_1}\right)^{(1-\nu)/\nu}\right] K_1 \tag{33b}$$

The first equation makes the determinant of the right-hand side of the system (29) zero. The second equation defines the kernel of the system, i.e. the combinations of capital stocks held by

[22] As an alternative, one could model a frictional labor market where free entry by firms determines employment in equilibrium (Berentsen, Menzio, and Wright, 2011; Rocheteau and Rodriguez-Lopez, 2014; Dong and Xiao, 2013).

households in states 0 and 1 that are possible solutions. Clearly, the ratios κ_0/μ_0 and κ_1/μ_1 are key. They represent Tobin's q, the marginal rate of substitution between using real income for investment or other purposes, for households in states 0 and 1. The second equation implies that if households are not to hold negative capital stocks, then the ratios κ_0/μ_0 and κ_1/μ_1 must be on opposite sides of 1. And according to Equation (25), households invest more than necessary to cover depreciation of their capital holdings if and only if their ratio κ/μ exceeds 1. Given what we have learned from the baseline version of the model, we can anticipate that households in state 1 will value money more relative to any other asset than households in state 0. Therefore, households in state 0 will indeed invest enough to increase their capital holdings, while households in state 1 will do the opposite, so that the overall capital stock remains in steady state.

If $\nu \to 0$, capital accumulation is frictionless. As we have seen, this is no problem for the existence and computation of steady states but it would make it impossible to solve for dynamics. Nevertheless, it provides a benchmark. It would still be the case that $\kappa_0/\mu_0 > \kappa_1/\mu_1$, i.e. households in state 0 would value capital more than households in state 1. Without the accumulation friction, Tobin's q must equal 1 for every agent choosing to accumulate capital. Consequently, we must have $\kappa_0/\mu_0 = 1 > \kappa_1/\mu_1$, so that only households in state 0 accumulate capital.

Definition 3. A strongly-monetary steady-state equilibrium with capital is a vector $\{c_0, c_1, h_0, h_1, \mu_0, \mu_1, \beta_0, \beta_1, \kappa_0, \kappa_1, q^B, \psi_0^B, \psi_1^B, q^K, \psi_0^K, \psi_1^K, n_1, Z_0, Z_1, B_1, K_0, K_1, R, \omega, T\}$ which satisfies Equations (27), (26), (24), (12), (13), and (30)-(33) with all the time derivatives equal to zero, and in which the two conditions (28) and $\omega h_0 - c_0 + T > 0$ hold.

The fully dynamic equilibrium is more complex, as was the case for the version of the model without physical capital. The price level in the extended model is determined by Equation (32), and expected inflation will not be equal to money growth along the transition path. The solution is to redo the construction from Equations (15) to (17). The latter equation stays unchanged:

$$\pi = \gamma - \frac{\dot{Z}_0 + \dot{Z}_1}{Z_0 + Z_1}$$

We need to differentiate Equation (32) with respect to time, substitute $\dot{c}_i$ and $\dot{h}_i$ using Equations (16), substitute the remaining time derivatives using the dynamic equations for the costate and aggregate state variables, and then use the result to substitute for $\dot{Z}_1$. We finally substitute $\dot{Z}_0$ using Equation (30a), and we are left with an equation describing expected inflation purely as a combination of contemporaneous variables. The definition of dynamic equilibrium is the obvious extension of Definitions 2 and 3.

4 Calibration

The next step is to calibrate the steady state of the full model with government intervention in the bond market and all three assets (money, government bonds, and physical capital) to the US economy. Households derive the following flow utility from consuming c units of the numéraire and spending h units of labor effort:

$$u(c,-h) = \theta \log(c) - \frac{\eta}{1+\tau} h^{1+\tau}, \qquad \theta, \eta, \tau \geq 0$$

The marginal utility of consuming the lumpy good d is normalized to one, and households discount the future at rate $\rho > 0$; in summary, preferences can be described by the parameters $\{\rho, \theta, \eta, \tau\}$.

The physical environment consists of the transition rates between states, the matching rates in the frictional asset markets, the production function, the rate at which physical capital depreciates, and the elasticity of investment. It can be described by the parameters $\{\varepsilon, \alpha, \chi^B, \chi^K, A, \xi, \delta, \nu\}$.

Government policy consists of the flow of transfers T, the bond supply B, the money growth rate γ, and the rates at which the government sells bonds in the frictional bond market, S. A steady state only exists if $S = 0$ to keep the bond supply unchanged, and the budget constraint makes one of the parameters T, B, or γ endogenous. Since there exist good calibration targets for B and γ, T is set to balance the budget in real terms. Consequently, policy in steady state is described by the parameters $\{B, \gamma\}$.

The unit of time is one year, so all rates are interpreted as annual, continuously compounded. Table 1 displays the calibration results. Some parameters correspond directly to a target, like γ to expected inflation, but others are jointly determined. In that case, they are displayed next to the target that most nearly determines them. Details of the parameter choice and a sensitivity analysis are provided in Appendix D.

The comparative statics of the calibrated model are illustrated in Figures 7-8, and can be summarized as follows. First, consider variations in the bond supply which are financed by lump-sum taxes. As calibrated, the bond price is in the interior region, with both buyers and sellers getting served with probability one. The lower bound on the real bond yield is 1.76% compared to a benchmark of 2%, and it would take only a 12% reduction in the bond supply to achieve this lower bound (i.e. the upper bound on the real bond price). This reduction in the bond supply would cause the capital-output ratio to increase, but only by an absolutely tiny amount: one-fifth of one percent of one percent. This crowding-in effect is accordingly dominated by the liquidity effect of the lower bond supply, causing output to fall 0.22% below the benchmark in steady state.[23]

[23] Unlike the other results discussed in this section, the estimate of the lower bound is very sensitive to the liquidity of capital, χ^K. How much of a premium households are willing to pay on bonds naturally depends on how good the alternatives are; for example, how liquid capital assets are and at what prices they trade. For the same reason, this

Description	Parameter	Target
Time preference	$\rho = 0.03$	Lucas (2000), also TIPS yields
Value of numéraire *	$\theta = 0.0458$	Interest elasticity of US money demand (Aruoba, Waller, and Wright, 2011)
Value of leisure	η	Cancels out with TFP parameter A
Labor supply elasticity	$\tau = 1$	Christiano, Trabandt, and Walentin (2010) (high end of range)
Liquidity shock *	$\varepsilon = 0.8354$	US money demand at 2% inflation (Aruoba, Waller, and Wright, 2011)
Goods market matching *	$\alpha = 3.8138$	US debt/GDP ratio ≈ 0.7
Bond market matching	$\chi^B = 52$	Trading at full surplus takes 1 week
Capital market matching	$\chi^K = 2$	Trading at full surplus takes 6 months
Capital share	$\xi = 0.36$	Francis and Ramey (2005)
Depreciation *	$\delta = 0.095$	Capital-output ratio of 2.88 (Francis and Ramey, 2005)
Investment elasticity	$\nu = 4.35$	Firm-level data (Jermann, 1998)
Supply of bonds	$B = 0.014$	US federal debt service payments
Expected inflation	$\gamma = 0.02$	Taylor rule inflation target

Table 1: Calibration parameters and targets, discussed in detail in Appendix D. * indicates parameters estimated jointly. The calibration implies $T/Y = -1.03\%$ (a small structural surplus).

On the other hand, the upper bound on the real bond yields is 3%, which would be attained with a 50% increase in the bond supply. The capital-output ratio would fall by two-fifths of one percent of one percent, but due to the liquidity effect output would increase by 1.00% above the benchmark in steady state. We can also interpret this upper bound on yields as saying that the world excess liquidity demand for US government debt would be satisfied at a US debt-to-GDP ratio of 105%, assuming that such an amount of debt could be supplied without raising doubt about solvency and financed with lump-sum taxes.

The qualitative behavior of the capital-output ratio matches the standard intuition that government bonds and private assets are substitutes; the value of capital as a saving vehicle is negatively related to the availability of the alternative, government bonds. However, reducing the bond supply (or increasing inflation) can be expansionary or contractionary because the overall direction of the capital stock and output are determined by two competing channels. If the labor supply was fixed, then an increase in the capital-output ratio would cause output to increase, but if the labor supply was very elastic, then an increase in the capital-output ratio could be consistent with a decrease in output. The quantitative result depends on two things: how elastic the labor supply is with respect

lower bound is also sensitive to inflation, which determines how much wealth households are willing to hold in the form of money.

to the marginal value of wealth, and how elastic the capital stock is to changes in its market value, which in turn depends on how abundant or scarce capital is compared to other assets. The calibration of my model suggests that in steady state, the liquidity channel strongly dominates the capital accumulation channel, so that *decreases in the bond supply are contractionary in the long run*.

We can also use this model to quantify the effect of inflation on output in steady state. Reducing inflation from its benchmark value of 2% to 0% raises output by 0.8% and increases real bond yields to 2.25%, because households will be willing to pay less of a premium to hold bonds. Increasing inflation to 4% will reduce output by 0.5% and leaves real bond yields almost unaffected, suggesting that the calibrated economy is close to the peak of the curve in Figure 9 (leftmost panel). This effect of anticipated inflation is not sensitive to the asset liquidity parameters, χ^B and χ^K, but it is very sensitive to the parameter τ which governs the elasticity of labor supply.[24]

As discussed earlier, the comparative statics of the bond supply and inflation rate depend critically on how these changes in government obligations are financed. In the previous analysis, I have assumed that lump-sum taxes adjust endogenously; now, let me assume instead that the monetary authority is willing to monetize a fixed real flow of deficits, $T + B = 0.37\%$ of output, and that the inflation rate varies endogenously. In this case, an increase in the bond supply by 50%, the upper bound of the interior region where further increases in the steady-state bond supply would cause bond sellers to be rationed in the inter-dealer market, would increase output by 1.02%. This is slightly more than in the case above, because now inflation falls to 1.9% as total demand for real balances increases. More interestingly, perhaps, a decrease of the bond supply all the way to zero would cause inflation to fall to 1.0%.

The reason is that trade in bond helps households concentrate real balances in the hands of those planning to use them. When such bond trade is not available, for example because the bond supply is zero, households hold a higher total of real balances. For a fixed inflow of real balances, the inflow relative to the total must fall, which shows up as lower money growth in the model but may not be so easily visible in the real world. This resembles a liquidity trap. Open market purchases of bonds 'normally' increase the quantity of money in circulation and cause higher prices, but when the bond supply is low (or bond demand is high), then central bank must buy the bonds from households who were really not planning to spend the money anytime soon, and who will therefore hold on to it. Not only will the temporary increase in money supply not be inflationary, the higher "money demand" in the economy reduces the relative rate of money growth, and thereby inflation, in the long run.

[24] A lower value for τ, such as the 0.1 which Christiano, Trabandt, and Walentin (2010) suggest as the lower bound of the reasonable range, would imply a much higher cost of steady-state inflation on output (see Appendix D).

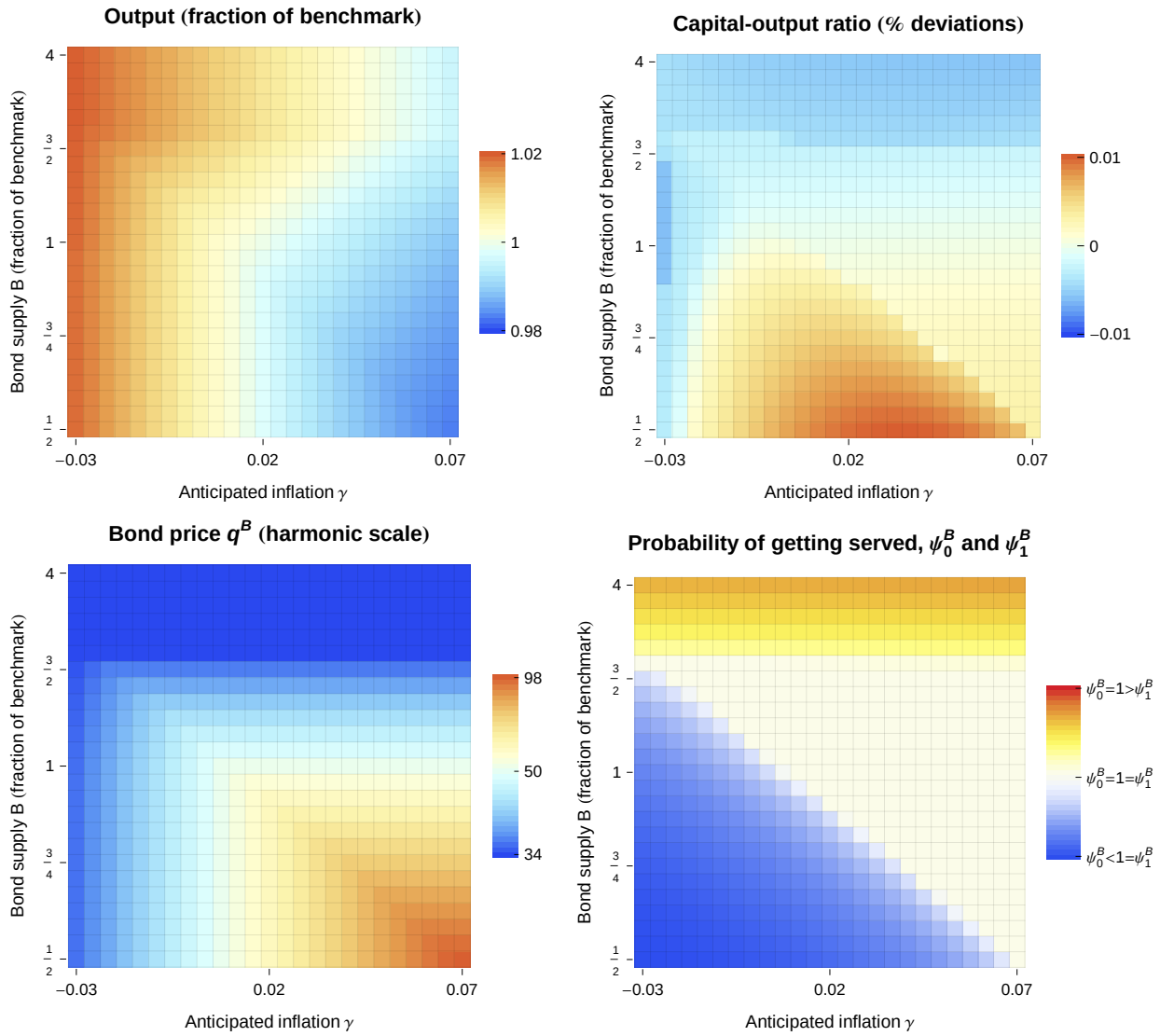


Figure 7: Comparative statics of the calibrated model with respect to the bond supply and the inflation rate, under the assumption that lump-sum transfers/taxes T are endogenous. The graph is three-dimensional: color represents the dependent variable (blue for low values, red for high ones).

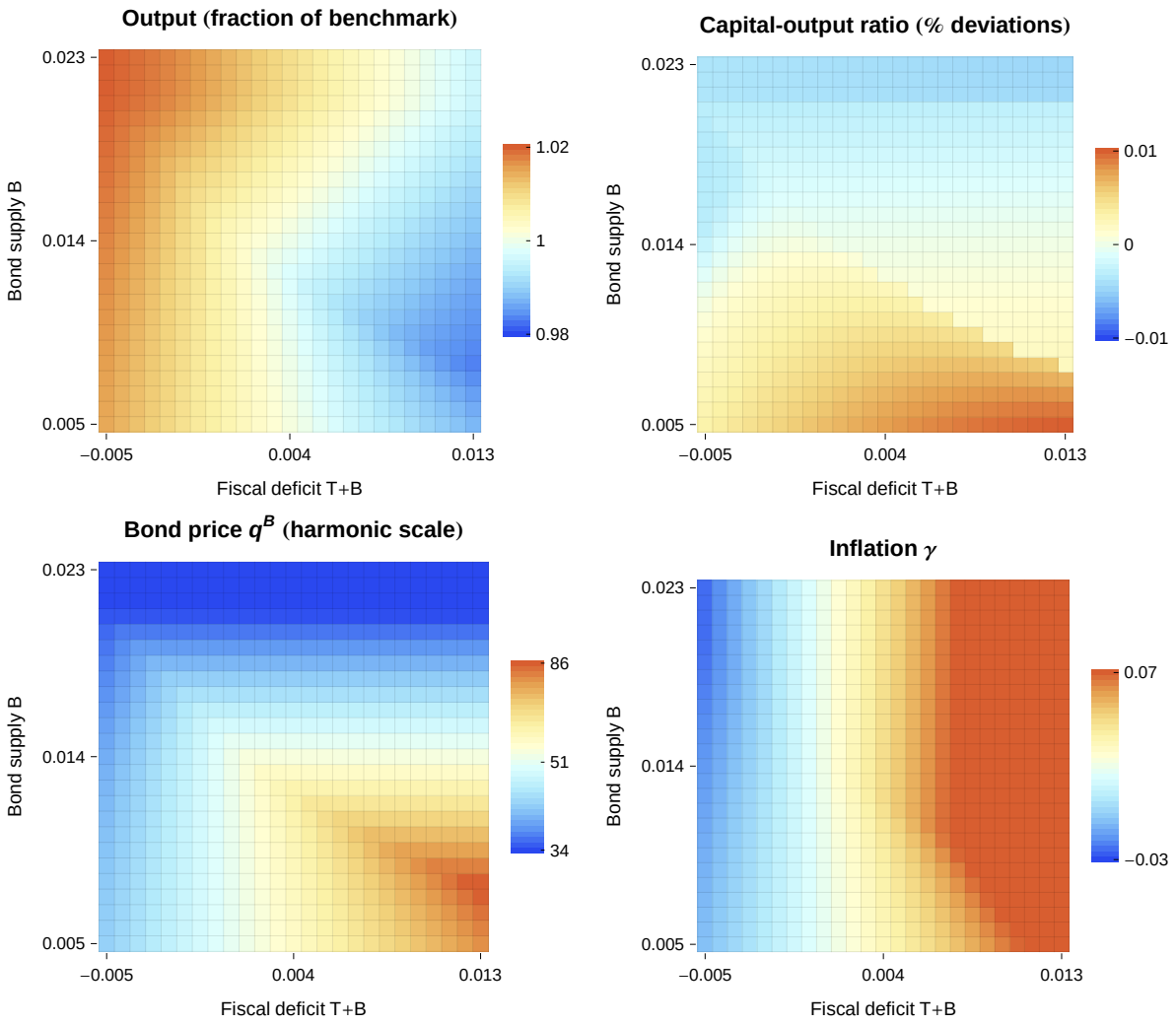


Figure 8: Comparative statics of the calibrated model with respect to the bond supply and the fiscal deficit, under the assumption that the inflation rate γ is endogenous. The graph is three-dimensional: color represents the dependent variable (blue for low values, red for high ones).

5 Conclusion

The model contributes to the theory of money and asset markets in important ways. For one, it is parsimonious: the only frictions are trading delays in asset markets and the fact that money is occasionally necessary to purchase consumption goods. Households are heterogeneous and their individual portfolios depend on history, but this is less of a "friction" than a feature. Financial assets are real and long-term, allowing us to abstract away from yield curve or inflation risk effects.[25] Even in this simple framework, money is not neutral in the short run, government intervention in asset markets has persistent effects, and the supply of illiquid assets matters for asset yields and for the macroeconomy.

A second advantage of the model is that it is highly tractable, flexible enough to accommodate a variety of extensions, and offers new insights into the pricing of illiquid assets. In contrast to the overwhelming majority of the literature, open-market operations can be modeled realistically as intervention in asset markets (rather than as directly manipulating households' budget constraints), and the difference matters. The liquidity channel and the capital accumulation channel can be cleanly distinguished theoretically and empirically (although future empirical work will be able to refine the estimation); while the literature on each channel in isolation is vast, to my knowledge the only papers to model both are Rocheteau and Rodriguez-Lopez (2014) and this paper.

The application of the model to the theory of quantitative easing yields the following results. Open-market purchases of imperfectly liquid long-term assets tend to reduce the yields on these assets and stimulate capital accumulation and output. However, there are several caveats to this conclusion. First, additional demand for financial assets by the central bank reallocates money from asset buyers to asset sellers; just as this stimulates private demand for goods, it crowds out private demand for assets in the short run, which will cause a period of higher yields and slower investment after the intervention has ended. Second, assets such as government bonds perform useful functions in this economy, and reducing their supply causes an efficiency loss that must be weighed against any stimulative effect. A calibration of the model suggests that this concern is realistic. Third, when assets are already scarce, further purchases can crowd out the private flow of funds. As a result, money is allocated less efficiently and the velocity of money is lower, which can cause disinflation and higher real interest rates, resembling a "liquidity trap".

The long-term effects of permanent open-market purchases are similar to those of higher long-run inflation, and they are a combination of two opposing forces. If the labor supply is very inelastic with respect to the marginal value of wealth, then these policies can stimulate capital accumulation

[25] Finite-term bonds can be liquidated in two ways: by selling them in the market, or by letting them mature. Since short-term bonds mature sooner than long-term bonds, they have inherently different liquidity properties. Geromichalos, Herrenbrueck, and Salyer (2016) analyze the implications of this fact for bond prices and the yield curve, and Williamson (2016) studies open market operations designed to "twist" the yield curve.

because capital is an alternative to money as a store of value. However, if the labor supply is elastic and capital is so abundant that it is not valued for its liquidity properties at the margin, then these policies reduce both the capital stock and output. One worry for advocates of QE should be that the U.S. capital stock ($\approx$ 300% of GDP) is indeed abundant relative to other stores of value such as Treasuries ($\approx$ 70%) or money ($\approx$ 15%, depending on the measure), and it is therefore likely that the liquidity premium on capital is small and inelastic. Naturally, looking at more liquid subcategories such as working capital or housing capital raises the likelihood that monetary policy could affect their prices, but in turn weakens the effect of those prices on the economy because a smaller sector is affected (unless perhaps there is a large intermediate goods multiplier).

An important qualification of the results is that as in most monetary-search models, higher output and capital accumulation do not always reflect higher welfare. In addition to the works cited earlier, Lagos and Rocheteau (2008) and Venkateswaran and Wright (2013) show that the liquidity value of capital may lead households to accumulate too much of it. In fact, the Friedman rule does achieve the first-best outcome in my model. However, the list of realistic model ingredients that could alter this result is extensive: if lump-sum taxes are not available (Hu, Kennan, and Wallace, 2009; Andolfatto, 2013), if the government has an advantage in providing certain goods and services and needs taxes to finance them, if a fraction of money is spent on socially worthless activity (e.g., crime; Williamson, 2012), or if some agents have market power, then the welfare properties of the model could be very different without affecting its ability to explain how government interventions in asset markets affect asset prices and the broader economy.

Acknowledgments

I am grateful to David Andolfatto, Paul Bergin, Martin Boileau, Francesca Carapella, Silvio Contessi, Joel David, William Dupor, Carlos Garriga, Athanasios Geromichalos, Andrew Glover, Christian Hellwig, Espen Henriksen, Kuk Mo Jung, Ricardo Lagos, Thomas Lubik, Fernando M. Martin, Miguel Molico, Christopher J. Neely, B. Ravikumar, Guillaume Rocheteau, Katheryn N. Russ, Kevin D. Salyer, Juan M. Sánchez, Ina Simonovska, Alan M. Taylor, Christopher J. Waller, David Wiczer, Russell Tsz-Nga Wong, and Randall Wright for their very useful comments and suggestions.

References

Alvarez, F., A. Atkeson, and P. J. Kehoe (2002, February). Money, interest rates, and exchange rates with endogenously segmented markets. *Journal of Political Economy 110*(1), 73–112.

Andolfatto, D. (2013). Incentive-feasible deflation. *Journal of Monetary Economics 60*(4), 383–390.

Aruoba, S. B., C. J. Waller, and R. Wright (2011). Money and capital. *Journal of Monetary Economics 58*(2), 98–116.

Ashcraft, A. and D. Duffie (2007). Systemic illiquidity in the federal funds market. *The American economic review 97*(2), 221–225.

Bauer, M. D. and C. J. Neely (2014). International channels of the fed's unconventional monetary policy. *Journal of International Money and Finance 44*, 24–46.

Baumol, W. J. (1952). The transactions demand for cash: An inventory theoretic approach. *The Quarterly Journal of Economics*, 545–556.

Berentsen, A., G. Camera, and C. Waller (2005). The distribution of money balances and the nonneutrality of money. *International Economic Review 46*(2), 465–487.

Berentsen, A., G. Camera, and C. Waller (2007). Money, credit and banking. *Journal of Economic Theory 135*(1), 171–195.

Berentsen, A., S. Huber, and A. Marchesiani (2014). Degreasing the wheels of finance. *International Economic Review 55*(3), 735–763.

Berentsen, A., G. Menzio, and R. Wright (2011). Inflation and unemployment in the long run. *American Economic Review 101*(1), 371–98.

Berentsen, A. and C. Waller (2011). Outside versus inside bonds: A modigliani-miller type result for liquidity constrained economies. *Journal of Economic Theory*.

Boldrin, M., L. J. Christiano, and J. D. M. Fisher (2001). Habit persistence, asset returns, and the business cycle. *The American Economic Review 91*(1), pp. 149–166.

Chiu, J. and M. Molico (2010). Liquidity, redistribution, and the welfare cost of inflation. *Journal of Monetary Economics 57*(4), 428 – 438.

Chiu, J. and M. Molico (2011). Uncertainty, inflation, and welfare. *Journal of Money, Credit and Banking 43*, 487–512.

Christiano, L. J., M. Trabandt, and K. Walentin (2010, June). Dsge models for monetary policy analysis. Working Paper 16074, National Bureau of Economic Research.

Cúrdia, V. and M. Woodford (2011). The central-bank balance sheet as an instrument of monetary policy. *Journal of Monetary Economics 58*(1), 54–79.

D'Amico, S. and T. B. King (2013). Flow and stock effects of large-scale treasury purchases: Evidence on the importance of local supply. *Journal of Financial Economics 108*(2), 425–448.

Del Negro, M., G. Eggertsson, A. Ferrero, and N. Kiyotaki (2011). The great escape? a quantitative evaluation of the fed's liquidity facilities. Technical report, Federal Reserve Bank of New York.

Dong, M. and S. X. Xiao (2013). Liquidity, monetary policy and unemployment. Technical report, mimeo.

Duffie, D., N. Gârleanu, and L. H. Pedersen (2005, November). Over-the-counter markets. *Econometrica 73*(6), 1815–1847.

Francis, N. and V. A. Ramey (2005). Is the technology-driven real business cycle hypothesis dead? shocks and aggregate fluctuations revisited. *Journal of Monetary Economics 52*(8), 1379 – 1399.

Fuerst, T. S. (1992). Liquidity, loanable funds, and real activity. *Journal of monetary economics 29*(1), 3–24.

Gagnon, J., M. Raskin, J. Remache, and B. Sack (2011). The financial market effects of the federal reserve's large-scale asset purchases. *International Journal of Central Banking 7*(1), 3–43.

Garbade, K. D. and J. F. Ingber (2005). The treasury auction process: Objectives, structure, and recent adaptations. *Current issues in economics and finance 11*(2).

Geromichalos, A. and L. Herrenbrueck (2016). Monetary policy, asset prices, and liquidity in over-the-counter markets. *Journal of Money, Credit, and Banking 48*(1), 35–79.

Geromichalos, A., L. Herrenbrueck, and K. Salyer (2016). A search-theoretic model of the term premium. *Theoretical Economics 11*(3), 897–935.

Geromichalos, A., J. M. Licari, and J. Suárez-Lledó (2007, October). Monetary policy and asset prices. *Review of Economic Dynamics 10*(4), 761–779.

Gertler, M. and P. Karadi (2011). A model of unconventional monetary policy. *Journal of Monetary Economics 58*(1), 17–34.

Greenwood, R. and D. Vayanos (2014). Bond supply and excess bond returns. *Review of Financial Studies 27*(3), 663–713.

He, Z. and A. Krishnamurthy (2013). Intermediary asset pricing. *The American Economic Review 103*(2), 732–70.

Herrenbrueck, L. and A. Geromichalos (2015). A tractable model of indirect asset liquidity. Working paper, Simon Fraser University.

Hu, T., J. Kennan, and N. Wallace (2009, 02). Coalition-proof trade and the friedman rule in the lagos-wright model. *Journal of Political Economy 117*(1), 116–137.

Huber, S. and J. Kim (2015). Central bank purchases of government bonds. *University of Zurich, Department of Economics, Working Paper* (193).

Jermann, U. J. (1998). Asset pricing in production economies. *Journal of Monetary Economics 41*(2), 257 – 275.

Krishnamurthy, A. and A. Vissing-Jorgensen (2012). The aggregate demand for treasury debt. *Journal of Political Economy 120*(2), 233–267.

Krishnamurthy, A. and A. Vissing-Jorgensen (2013). The ins and outs of lsaps. Manuscript.

Lagos, R. (2010, November). Asset prices and liquidity in an exchange economy. *Journal of Monetary Economics 57*(8), 913–930.

Lagos, R. (2011, October). Asset prices, liquidity, and monetary policy in an exchange economy. *Journal of Money, Credit and Banking 43*, 521–552.

Lagos, R. and G. Rocheteau (2008, September). Money and capital as competing media of exchange. *Journal of Economic Theory 142*(1), 247–258.

Lagos, R. and G. Rocheteau (2009). Liquidity in asset markets with search frictions. *Econometrica 77*(2), 403–426.

Lagos, R. and R. Wright (2005, June). A unified framework for monetary theory and policy analysis. *Journal of Political Economy 113*(3), 463–484.

Lagos, R. and S. Zhang (2015). Monetary exchange in over-the-counter markets: A theory of speculative bubbles, the fed model, and self-fulfilling liquidity crises.

Lester, B., A. Postlewaite, and R. Wright (2011, October). Information and liquidity. *Journal of Money, Credit and Banking 43*, 355–377.

Li, Y. and G. Rocheteau (2011, April). On the threat of counterfeiting. *Macroeconomic Dynamics 15*(S1), 10–41.

Lucas, R. (2000). Inflation and welfare. *Econometrica 68*(2), 247–274.

Lucas, R. and E. C. Prescott (1971). Investment under uncertainty. *Econometrica 39*(5), pp. 659–681.

Mattesini, F. and E. Nosal (2015). Liquidity and asset prices in a monetary model with otc asset markets. *Journal of Economic Theory*.

Nosal, E. and G. Rocheteau (2011). *Money, payments, and liquidity*. The MIT Press.

Nosal, E. and N. Wallace (2007, May). A model of (the threat of) counterfeiting. *Journal of Monetary Economics 54*(4), 994–1001.

Robertson, D. (1940). *Essays in Monetary Theory*. London: P.S. King and Son.

Rocheteau, G. (2009). Information and liquidity: a discussion. Working Paper 0902, Federal Reserve Bank of Cleveland.

Rocheteau, G. (2011). Payments and liquidity under adverse selection. *Journal of Monetary Economics 58*(3), 191–205.

Rocheteau, G. and A. Rodriguez-Lopez (2014). Liquidity provision, interest rates, and unemployment. *Journal of Monetary Economics 65*, 80–101.

Rocheteau, G., P.-O. Weill, and T.-N. Wong (2015). A tractable model of monetary exchange with ex-post heterogeneity. Technical report, National Bureau of Economic Research.

Rocheteau, G. and R. Wright (2012). Liquidity and asset-market dynamics. *Journal of Monetary Economics*.

Thornton, D. L. (2012). Evidence on the portfolio balance channel of quantitative easing. Technical report, Federal Reserve Bank of St. Louis.

Tobin, J. (1956). The interest-elasticity of transactions demand for cash. *The Review of Economics and Statistics 38*(3), 241–247.

Tobin, J. (1965). Money and economic growth. *Econometrica: Journal of the Econometric Society*, 671–684.

Trejos, A. and R. Wright (2011). Money and finance: An integrated approach. Manuscript.

Vayanos, D. and P.-O. Weill (2008). A search-based theory of the on-the-run phenomenon. *The Journal of Finance 63*(3), 1361–1398.

Venkateswaran, V. and R. Wright (2013, December). Pledgability and liquidity: A new monetarist model of financial and macroeconomic activity. In *NBER Macroeconomics Annual 2013, Volume 28*, NBER Chapters. National Bureau of Economic Research, Inc.

Williamson, S. D. (2006, 02). Search, limited participation, and monetary policy. *International Economic Review 47*(1), 107–128.

Williamson, S. D. (2012, October). Liquidity, monetary policy, and the financial crisis: A new monetarist approach. *American Economic Review 102*(6), 2570–2605.

Williamson, S. D. (2016). Scarce collateral, the term premium, and quantitative easing. *Journal of Economic Theory 164*, 136–165.

Williamson, S. D. and R. Wright (2010, October). New monetarist economics: Models. In B. M. Friedman and M. Woodford (Eds.), *Handbook of Monetary Economics*, Volume 3 of *Handbook of Monetary Economics*. Elsevier.

APPENDIX

A Proofs

Proof. Proof of Lemma 1. In order to prove the statement, we need to find the saddle-path solution $\{\mu_0^s(t), \mu_1^s(t), \beta_0^s(t), \beta_1^s(t)\}$. I claim that the saddle path solution is the one that satisfies $\dot{\mu}_i = \dot{\beta}_i = 0$ for $i = 0, 1$. The costate equations (4) do not depend on the individual state variables z or b. If, as assumed, $-T(t) < h_i(t) - c_i(t)$, then $\dot{z}_i > 0$ for any combination of asset holdings, and since also $\pi(t) > -\rho$, any path not satisfying $\dot{\mu}_i = \dot{\beta}_i = 0$ would imply exploding paths for μ_i and hence violate the transversality conditions. (The claim does in fact not hold if $-T(t) > h_i(t) - c_i(t)$ at any point in time, as the saddle path for a household intending to let its money holdings run down to zero looks very different, a perhaps surprising asymmetry.)

The solution to the optimal control problem satisfies:

$$\frac{\partial}{\partial z} W_0(z,b) = \mu_0^s \qquad \frac{\partial}{\partial b} W_0(z,b) = \beta_0^s$$
$$\frac{\partial}{\partial z} W_1(z,b) = \mu_1^s \qquad \frac{\partial}{\partial b} W_1(z,b) = \beta_1^s$$

And as shown above, the saddle-path solution $\{\mu_0^s(t), \mu_1^s(t), \beta_0^s(t), \beta_1^s(t)\}$ does not depend on z or b. The statement follows. □

Proof. Proof of Lemma 2. Take first-order conditions of the maximization problems (1) and (2) and use Equations (34). The statement follows. □

Proof. Proof of Proposition 1. Focus on steady states, and begin with assuming that $\chi = 0$. In that case, $\mu_0 < \mu_1 < 1$ is obvious given that $\pi > -\rho$. Next, add the equations for β_0 and β_1 and arrange them to yield:

$$(\rho + \alpha + \varepsilon)(\beta_1 - \beta_0) = \mu_1 - \mu_0,$$

establishing $\beta_0 < \beta_1$. Now, the equations for β_0 and β_1 can easily be solved for in terms of μ_0 and μ_1, and then arranged to yield:

$$\frac{\beta_0}{\mu_0} = \frac{\rho + \alpha + \varepsilon\mu_1/\mu_0}{\rho(\rho + \varepsilon + \alpha)}$$
$$\frac{\beta_1}{\mu_1} = \frac{\alpha\mu_0/\mu_1 + \rho + \varepsilon}{\rho(\rho + \varepsilon + \alpha)}$$

As $\mu_0 < \mu_1$, two things follow: first, $\beta_0 < \beta_1$, and second, the claim that $\beta_0/\mu_0 > \beta_1/\mu_1$. So the guess that households in state 0 act as buyers of bonds and households in state 1 act as sellers has

been verified for the case $\chi = 0$. Now, consider a small increase in χ; clearly, the inequalities must be preserved. Finally, let χ increase to infinity. In that case, the equations converge to:

$$\mu_0 = \beta_0/q \qquad \rho\beta_0 = \mu_0 + \varepsilon(\beta_1 - \beta_0)$$
$$(\rho+\pi)\mu_1 = \alpha(1-\mu_1) \qquad \beta_1 = \mu_1 q$$

So in the limit, clearly $\beta_0/\mu_0 = q = \beta_1/\mu_1$, indicating that the inequality holds for any $\chi \in [0,\infty)$.

Now turn to the remaining two inequalities. Consider the equation for β_0 and rearrange it:

$$\rho\beta_0 = \mu_0 + \varepsilon(\beta_1 - \beta_0) \qquad \Rightarrow \qquad \rho\frac{\beta_0}{\mu_0} = 1 + \varepsilon\frac{\beta_1 - \beta_0}{\mu_0}$$

This equation does not depend on χ. As $\beta_0 < \beta_1$ (proven above), the claim $\beta_0/\mu_0 > 1/\rho$ follows.

Finally, assume that $q = \beta_1/\mu_1$, consider the equation for β_1, and rearrange it:

$$\rho\beta_1 = \mu_1 + \alpha(\beta_0 - \beta_0) \qquad \Rightarrow \qquad \rho\frac{\beta_1}{\mu_1} = 1 - \alpha\frac{\beta_1 - \beta_0}{\mu_1}$$

As $\beta_0 < \beta_1$, the claim $\beta_1/\mu_1 < 1/\rho$ follows: when bonds are priced at the lowest level possible (i.e. when they are abundant), then their price reflects an illiquidity discount. □

Sufficient conditions for $h_0 - c_0 + T > 0$. This condition is required for a tractable solution, because without it, some state-0 households would want to decumulate money balances rather than accumulate them. It can be shown that decumulation leads to non-linear value functions, rendering the model intractable. Unfortunately, there is no simple restriction on exogenous parameters equivalent to $h_0 - c_0 + T > 0$, but there are some sufficient conditions.

For example, notice that the government budget constraint in steady state is $T = \gamma(Z_0 + Z_1) - B$. Therefore, we need $B < h_0 - c_0 + \gamma(Z_0 + Z_1)$. Suppose $\gamma \geq 0$ so that there is no deflation that would have to be financed with tax revenue, then $B < h_0 - c_0$ is sufficient. How large can we allow the debt service flow B (and the implied flow of lump-sum taxes T) to get? Fix all parameters other than B and T. Then the term $h_0 - c_0$ achieves its minimum (call it $h_0^{\min} - c_0^{\min}$) for $B \to 0$ and $q \to \beta_0/\mu_0$ (the inflation tax is most keenly felt when there are no saving vehicles other than money). In this case, $\mu_1 = \alpha/(\rho+\gamma+\alpha)$ and $\mu_0 = \varepsilon\alpha/(\rho+\gamma+\varepsilon)/(\rho+\gamma+\alpha)$. We can then use Equations (6) to solve for $h_0^{\min}$ and $c_0^{\min}$. The simplest sufficient condition for existence of a strongly-monetary equilibrium is thus $B < h_0^{\min} - c_0^{\min}$, where the right-hand-side depends on $\{u, \rho, \varepsilon, \alpha, \gamma\}$ alone.

If instead $\gamma < 0$, so that there is deflation which must be financed with taxes, we must solve for Z_0 and Z_1, too, which can get complex even if a closed form solution exists. But suppose $\gamma \to -\rho$, so we are approaching the Friedman Rule in the limit and asset valuations become trivial: $\mu_0 = \mu_1 = 1$ and $\beta_0 = \beta_1 = 1/\rho$. The rate of money accumulation $h - c$ is now the maximal

possible level $h^* - c^*$, which is defined through Equations (6) together with $\mu_0 = \mu_1 = 1$.

Now, consider two subcases which we can solve completely: first, let the bond supply be zero, and second, let it be such that both buyers and sellers in the decentralized bond market always get to trade. That is, $\psi_0 = \psi_1 = 1$; notice that as we approach the Friedman Rule, the set of bond supplies for which equilibrium is in this intermediate region shrinks to a singleton.

In the first case, $B = 0$ implies $\psi_0 = 0$ and $\psi_1 = 1$, so we can ultimately solve for:

$$-\frac{T}{h^* - c^*} = \frac{\rho(\alpha^2 + \alpha\varepsilon + \varepsilon^2 - (\alpha+\varepsilon)\rho)}{\alpha\varepsilon(\alpha+\varepsilon-\rho)}$$

We want this term to be less than one; it is easy to verify that this is the case for $\rho < \min\{\varepsilon, \alpha\}$.

And in the second case, where the bond supply is exactly such that both bond buyers and sellers trade with certainty ($\psi_0 = \psi_1 = 1$), we can solve for:

$$-\frac{T}{h^* - c^*} = \frac{\rho\left[2\alpha^3 + \alpha^2(2\varepsilon + 3\chi - 2\rho) + [\varepsilon\chi + \alpha(\varepsilon+\chi)](\varepsilon+\chi-2\rho)\right]}{\alpha\left[\alpha\varepsilon(\alpha+\varepsilon-\rho) + \chi[(\alpha+\varepsilon)^2 - \varepsilon\rho] + \chi^2(\alpha+\varepsilon)\right]}$$

Again, we want this term to be less than one; one can verify that this is the case for $\rho < \min\{\varepsilon, \alpha, \chi\}$.

If none of these special cases apply, the condition $h_0 - c_0 + T$ must be verified numerically for a candidate equilibrium.

B Comparative statics of inflation

The discussion in Section 2.6 has already suggested that the comparative statics of inflation in a steady-state equilibrium of the baseline model are important, not least because they are entwined with the comparative statics of bond supply. To begin with, recall that $\beta_0/\mu_0 > \beta_1/\mu_1$ (relative to money, households in state 0 value bonds more than households in state 1). It is also true that as a function of the bond inter-dealer market price q, both β_0/μ_0 and β_1/μ_1 are increasing. Consequently, the highest possible price of bonds in terms of real balances, given all other parameters affecting asset valuations, is obtained by solving the Euler equations (4) together with $q = \beta_0/\mu_0$; denote this price by $\overline{q}$. Similarly, the lowest possible price of bonds is obtained by solving the Euler equations with $q = \beta_1/\mu_1$; denote this price by $\underline{q}$.

Lemma 3. *For a given set of parameters $(\rho, \varepsilon, \alpha, \chi)$, the bond price bounds satisfy:*

1. *If $\pi > -\rho$, $\underline{q} < 1/\rho < \overline{q}$;*
2. *As $\pi \to -\rho$, $\underline{q} \to 1/\rho$ and $\overline{q} \to 1/\rho$;*
3. *As π increases, $\underline{q}$ strictly decreases and $\overline{q}$ strictly increases;*

4. $\overline{q} < \infty$ *if and only if* $\pi\chi < \rho(\rho+\varepsilon+\alpha)$.

Proof. Setting $q = \beta_1/\mu_1$, and again beginning with $\chi = 0$, we can solve:

$$\underline{q} = \frac{\beta_1}{\mu_1} = \frac{(\rho+\pi+\varepsilon)(\rho+\varepsilon+\alpha)-(\rho+\pi)\alpha}{\rho(\rho+\pi+\varepsilon)(\rho+\varepsilon+\alpha)} < \frac{1}{\rho}$$

As $\chi \to \infty$, we have $\beta_1/\mu_1 \to 1/\rho$; so the claim holds for every $\chi \in [0,\infty)$.

Now setting $q = \beta_0/\mu_0$, we can solve:

$$\overline{q} = \frac{\beta_0}{\mu_0} = \frac{(\rho+\pi)+(\rho+\varepsilon+\alpha+\chi)}{\rho(\rho+\varepsilon+\alpha)-\pi\chi}$$

Clearly, $\overline{q} > 1/\rho$, and $\overline{q} < \infty$ if and only if $\pi\chi < \rho(\rho+\varepsilon+\alpha)$.

And as $\pi \to -\rho$, we must have $\mu_0 \to \mu_1$, hence $\beta_0 \to \beta_1$, and consequently, $\underline{q} \to 1/\rho$ and $\overline{q} \to 1/\rho$ for any χ. □

So in principle, as inflation increases, the market price for bonds could increase or decrease as the range of acceptable terms of trade widens. But it is Z_0/B_1, the ratio of bond demand to supply, that determines whether the price is at either boundary or interior. As inflation does not affect the steady-state value of B_1 in the interior region where $\psi_1 = 1$, it is the effect of inflation on the real balances held by households in state 0 that determines the behavior of the bond price. As it turns out, Z_0 is either strictly decreasing or hump-shaped as a function of inflation. Starting at the Friedman rule $\pi = -\rho$, a small increase in π has two effects: it reduces the rate at which households in state 0 accumulate money, but through the inflation tax, it redistributes money from those who hold more on average (households in state 1) to those who hold less on average (households in state 0). Either effect could be stronger. But as π becomes very large, all households hold few real balances, so Z_0 necessarily tends to zero.

The fact that $\overline{q} \to \infty$ for a finite inflation rate means that, in an equilibrium in which buyers of bonds are rationed, potential buyers (households in state 0) are willing to hold real bonds with a zero rate of return, because avoiding the inflation tax is such a valuable service. As bond buyers would be willing to pay any price in the frictional asset market, they could never be rationed. Therefore, the region where bond buyers are rationed does not exist in this case, and the demand curve for bonds asymptotically approaches the vertical axis.

As Figure 9 shows, the level of bond supply plays an essential role in determining the comparative statics of inflation. When B is low (top row), then the price of bonds tends to be at the lower bound $\underline{q}$, which is decreasing in inflation. The hump-shaped effect of inflation on bond demand, however, may make the price of bonds non-monotonic, too. When B is high (bottom row), by contrast, the price of bonds tends to be at the upper bound $\overline{q}$, which is increasing in inflation. As

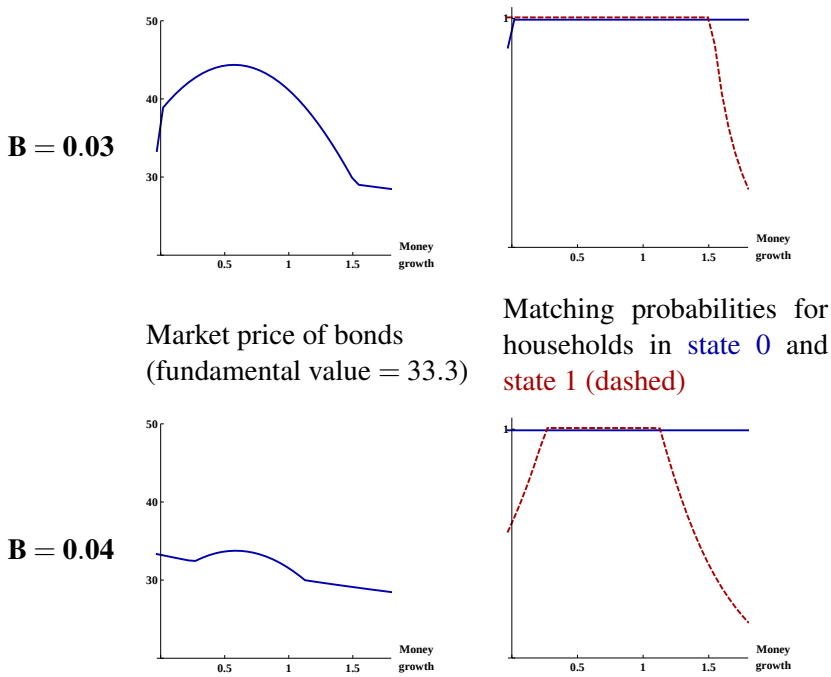


Figure 9: Comparative statics of inflation. Key parameters: $\rho = .03$, $\varepsilon = .5$, $\alpha = 1$, $\chi = 6$.

Z_0 cannot become arbitrarily large, the bond price is necessarily non-monotonic in inflation in this case. However, as the figure also makes clear, the non-monotonic effects of inflation may well require very high rates of inflation, on the same order of magnitude as the arrival rates of the state transitions (i.e. the liquidity shocks).

The comparative statics of bond liquidity, represented by the rate of matching in the decentralized asset market χ, can be described as follows. If both inflation and bond liquidity are so high that $\chi\pi \geq \rho(\rho+\varepsilon+\alpha)$, then $\overline{q}$ is infinite. The region where buyers are rationed does not exist, and the demand curve for bonds asymptotically approaches the vertical axis. If $\chi\pi < \rho(\rho+\varepsilon+\alpha)$, by contrast, then $\overline{q} < \infty$, so the price of bonds is bounded from above. The bond price bounds $\overline{q}$ and $\underline{q}$ are increasing in χ, so if either side of the market is rationed in equilibrium, then an increase in the rate of matching increases the price of bonds. The same is not necessarily true in the interior region, however; Z_0 and B_1 are both equilibrium objects, and their ratio may increase or decrease as the rate of matching increases. Put another way, an increase in bond liquidity will reduce yields if the bonds are either abundant or scarce, but it may increase yields in an interior region.

C Brokers have some market power

Based on the canonical Duffie, Gârleanu, and Pedersen (2005), assume that when matched, households and brokers determine the size of the trade and the split of the surplus by Nash bargaining,

where the broker has a "bargaining power" exponent of $\zeta \in [0,1]$. The model of the main text is obtained for $\zeta = 0$.

In principle, brokers could use their bargaining power to claim profits in terms of either money or bonds. For now, assume that brokers claim profits by retaining money alone, so that market clearing will be expressed by equality of the supply and demand for bonds (although there is some small loss of generality). The supply of money will equal the demand for money plus profits retained, and the brokers will instantly remit the profits to their owners.

First, consider a match between a household with real balances z and bonds b, and a broker with access to the inter-dealer market at price q. Denote by b_M the quantity of bonds bought by the household (sold if negative), and by q_M the "match-specific price" charged or offered for those bonds. Guess ahead that the value function of money and bonds is again linear, with a marginal value of real balances μ and a marginal value of bonds β, and assume that the owners of the brokerage firm also have a linear valuation of money. Then the Nash bargaining solution satisfies:

$$(b_M, q_M) \in \underset{b_M, q_M}{\arg\max} \left\{ (-\mu q_M b_M + \beta b_M)^{1-\zeta} (q_M b_M - q b_M)^{\zeta} \right\}$$
$$\text{subject to } b_M \in [-b, z/q_M]$$

The bargaining solution has two cases, corresponding to whether the household buys or sells bonds. The first case, where the household is a buyer, obtains if $\beta/\mu > q$:

$$b_M = z/q_M \quad \text{and} \quad q_M = \left[\zeta \left(\frac{\beta}{\mu} \right)^{-1} + (1-\zeta) q^{-1} \right]^{-1}$$

So the household will spend all of its money to buy bonds, and the ask price charged is the *harmonic mean* of the parties' marginal rates of substitution, weighted by their bargaining power. The second case, where the household is a seller, obtains if $\beta/\mu < q$:

$$b_M = -b \quad \text{and} \quad q_M = \zeta \frac{\beta}{\mu} + (1-\zeta) q$$

So the household will sell all of its bonds, and the bid price offered is the *arithmetic mean* of the parties' marginal rates of substitution, weighted by their bargaining power.

In equilibrium, because $\beta_0/\mu_0 > \beta_1/\mu_1$, households in state 0 will buy bonds at the ask price and households in state 1 will sell bonds at the bid price when matched with a broker. Substituting the results into the households' Euler equations, these equations take the following form:

$$\rho \mu_0 = \dot{\mu}_0 - \pi \mu_0 + \varepsilon [\mu_1 - \mu_0] + \chi (1-\zeta) \left[\frac{\beta_0}{q} - \mu_0 \right] \tag{35a}$$

$$\rho\mu_1 = \dot{\mu}_1 - \pi\mu_1 + \alpha[1-\mu_1] \tag{35b}$$

$$\rho\beta_0 = \dot{\beta}_0 + \mu_0 + \varepsilon[\beta_1 - \beta_0] \tag{35c}$$

$$\rho\beta_1 = \dot{\beta}_1 + \mu_1 + \alpha[\beta_0 - \beta_1] + \chi(1-\zeta)\,[q\mu_1 - \beta_1] \tag{35d}$$

We can see that as far as households' decision making is concerned, giving brokers bargaining power $\zeta > 0$ is equivalent to increasing the trading delay. General equilibrium is more complicated because brokers now have profits that must be transmitted to households, and because it is no longer the inter-dealer market price that clears the inflow and outflow of both money and bonds. If brokers on both sides of the market claim their profits in retained money, then the inflow and the outflow of bonds have to be equal. The ask price (not the inter-dealer price) has to equal Z_0/B_1 in the interior region, and the lower bound on the ratio Z_0/B_1 that determines whether sellers are rationed is no longer equal to the marginal rate of substitution of the sellers, β_1/μ_1, but it is equal to the ζ-weighted harmonic mean of β_0/μ_0 and β_1/μ_1. If brokers claimed their profits in retained bonds instead, things would be different. All this shows that brokers having market power complicates the model considerably, and without adding much insight for the main purpose of the paper: understanding monetary intervention in illiquid markets.

D Details of the calibration, and robustness

The branch of the business cycle literature studying models with capital adjustment costs suggests targets for some of my parameters: $\rho = 0.04$, $\delta = 0.085$ (in annual, continuously compounded terms), $\xi = 0.36$, and $\nu = 1/0.23 = 4.35$ (Jermann, 1998; Boldrin, Christiano, and Fisher, 2001; Francis and Ramey, 2005). Following Lucas (2000), however, I prefer $\rho = 0.03$; indeed, long-term TIPS yields in the US have never exceeded 3%. In the benchmark calibration, the average yield on government bonds is $1/q^B = 2\%$; as the fundamental yield is $\rho = 3\%$, the bonds do carry a liquidity premium.

In my model, households generally value capital at a premium; as a consequence, the appropriate target for the depreciation rate is not the δ from the business cycle literature, but the capital-output ratio. In steady state, it must satisfy $K/Y = \xi/R$, so I set $K/Y = 2.88$ (backed out from Francis and Ramey, 2005) to determine $R = 0.125$, and then compute the appropriate δ. Because the capital-output ratio is so high relative to the debt-output ratio (which targets $q^B B/Y = 0.7$), the premium on capital implied by the model is tiny.

It is not obvious which empirical targets most closely correspond to the asset market matching rates χ^B and χ^K. Trade volume in asset markets could be a plausible target, but in the model there is only one motive for trading—liquidity shocks—while in reality, there are many more. Another option is to match the spread between the price that bonds would fetch if issued in the competitive

market for fruits and money (β_0/μ_0 in the model) and the secondary market price (q^B) to the off-the-run discount observed in Treasury markets. However, Vayanos and Weill (2008) estimate that this spread was 30-60 basis points in the 1990s and early 2000s; even allowing that overall yields were higher then, this strategy would imply *extremely* illiquid markets in my model. Indeed, in the benchmark calibration with $\chi^B = 52$, the implied spread between primary and secondary yields is only *a quarter of a basis point*. This tiny difference is explained by the fact that the model only allows for very limited heterogeneity in preferences, liquidity needs, or beliefs, and no differences in bond maturity. But it does illustrate that even US Treasuries often trade in markets with a fair amount of illiquidity.

Rather than pretending to match any single feature of reality exactly while leaving out a million others, the more honest approach would be to start with a fair guess for the χ's, discuss whether the guess is reasonable, and examine whether the results are robust. For robustness, I present below a version of the calibration with $\chi = 17.33$, a third of the benchmark value.

	Benchmark	Low χ^B	Low τ	Low χ^B and τ
Time preference	$\rho = 0.03$			
Capital market matching	$\chi^K = 2$			
Capital share	$\xi = 0.36$			
Depreciation *	$\delta = 0.095$			
Investment elasticity	$\nu = 4.35$			
Supply of bonds	$B = 0.014$			
Expected inflation	$\gamma = 0.02$			
Labor supply elasticity	$\tau = 1$		$\tau = 0.316$	
Bond market matching	$\chi^B = 52$	$\chi^B = 17.33$	$\chi^B = 52$	$\chi^B = 17.33$
Value of numéraire *	$\theta = 0.0811$	$\theta = 0.0177$	$\theta = 0.0458$	$\theta = 0.0033$
Liquidity shock *	$\varepsilon = 0.7918$	$\varepsilon = 0.9640$	$\varepsilon = 0.8354$	$\varepsilon = 0.9845$
Goods market matching *	$\alpha = 3.6002$	$\alpha = 4.3188$	$\alpha = 3.8138$	$\alpha = 4.4204$

Table 2: Alternative calibration parameters. * indicates parameters estimated jointly. Depreciation of capital was also estimated jointly, but it did not vary up to 6 decimals.

And to judge whether the liquidity parameters are reasonable, note that for a Poisson process with arrival rate χ, the expected time until a match arrives is $1/\chi$. Consequently, $\chi^B = 52$ and $\chi^K = 2$ is equivalent to saying that it takes one week on average to complete a trade in the bond market, and six months to complete a trade in the capital market, *at full surplus*. If the brokers have some market power (see Appendix C), then the true matching rate is discounted by the share of the surplus claimed by the household. For example, the benchmark $\chi^K = 2$ could be interpreted as saying that it takes only three months to sell capital, but that intermediaries will claim half of the surplus generated. As for government bond markets: yes, they are known to be highly liquid,

but not perfectly so, and it is actually very difficult to trade these bonds *at full surplus*. After all, it is the business model of all the intermediaries that make trade so fast in these markets that trade would be much slower without them.

Robustness with respect to lower bond market liquidity χ^B

The calibrated parameters using a lower value of χ^B are shown in the second and fourth column of Table 2. While some of the parameters have changed, the key comparative statics results did not change much, as shown in the second and fourth column of Table 3.

	Benchmark	Low χ^B	Low τ	Low χ^B and τ
Lower bound bond yield	1.77	1.76	1.75	1.75
Change in the bond supply required to achieve this lower bound	-12	-12	-14	-14
Resulting change in output	-0.24	-0.22	-0.08	-0.72
Resulting change in K/Y	+0.002	+0.002	+0.003	+0.002
Upper bound bond yield	2.94	3.01	2.94	3.01
Change in the bond supply required to achieve this upper bound	+50	+53	+.53	+56
Resulting change in output	+1.00	+0.94	+3.02	+2.90
Resulting change in K/Y	-0.004	-0.003	-0.005	-0.003
Real bond yield with 0% inflation	2.17	2.25	2.14	2.24
Effect on output of 0% inflation	+0.8	+0.8	+2.1	+2.4
Real bond yield with 4% inflation	2.00	2.00	2.03	2.02
Effect on output of 4% inflation	-0.6	-0.5	-1.6	-1.6

Table 3: Implications of the alternative calibrations, all numbers are **in percent**. Each calibration includes 2% inflation and a 2% yield on the real consol bond as targets.

The main reason why the calibrated results are so similar even with big differences in bond liquidity is that a more liquid bond market does not necessarily affect the bond price very much when it is in the intermediate region: faster trade reduces the misallocation of portfolios, i.e. it reduces both Z_0 and B_1, and the ratio of the two may change in either direction. Furthermore, the bond supply and the bond yield are both calibration targets, so what adjusts to the changing bond liquidity target are the relative utility of the lumpy consumption good and the numéraire, and the frequency of the liquidity shocks. The end result is almost identical comparative statics.

Robustness with respect to lower labor supply elasticity τ

The first-order condition for labor effort of a household in state i is $h_i = [(\mu_i \omega)/\eta]^{1/\tau}$, where μ_i is the marginal utility of real balances, ω is the real wage, and η shifts the value of leisure.

The parameter τ governs the elasticity of labor supply both with respect to the real wage and with respect to the marginal utility of wealth; but in this model, the marginal utility of wealth equals the marginal utility of money. Therefore, the higher τ is, the less sensitive is the aggregate supply of labor to inflation, and the weaker is the effect of the inflation tax on labor supply and output. If we assume for a moment that assets other than money are not priced for their liquidity (i.e. they are either illiquid or abundant), then the following simple relationships hold:

$$\mu_0 = \frac{\varepsilon \cdot \mu_1}{\rho+\gamma+\varepsilon}, \quad \mu_1 = \frac{\alpha}{\rho+\gamma+\alpha}, \quad \text{and} \quad H = \left(n_0\mu_0^{1/\tau} + n_1\mu_1^{1/\tau}\right)\left(\frac{\omega}{\eta}\right)^{1/\tau}$$

If we further consider that for low inflation $\mu_0 \approx \mu_1$, we can derive the following approximation for the semi-elasticity of labor supply with respect to inflation:

$$\begin{aligned}
\frac{\mathrm{d}\log(H)}{\mathrm{d}\gamma} &\approx \frac{1}{\tau}\left(n_0\frac{\mathrm{d}\log(\mu_0)}{\mathrm{d}\gamma} + n_1\frac{\mathrm{d}\log(\mu_1)}{\mathrm{d}\gamma}\right) \\
&\approx \frac{1}{\tau}\left(\frac{n_0}{\rho+\gamma+\varepsilon} + \frac{n_0+n_1}{\rho+\gamma+\alpha}\right) \\
&\approx \frac{1}{\tau}\left(\frac{\alpha}{\varepsilon(\varepsilon+\alpha)} + \frac{1}{\alpha}\right)
\end{aligned}$$

The calibration for ε and α does not depend much on τ, and the term in parentheses evaluates to approximately 1 for all of the calibrations. So to a rough approximation, the semi-elasticity of labor supply with respect to inflation is $1/\tau$.

The question which choice for $1/\tau$ represents the best estimate of the elasticity of aggregate labor supply with respect to the real wage is controversial; Christiano, Trabandt, and Walentin (2010), for example, hedge their bets by considering values of $1/\tau = 1$ and 10. In my model, the same parameter also has to represent the elasticity of labor supply with respect to the valuation of *fiat money*, which is likely to be lower because most households do not hold their marginal wealth in money. Consequently, in my benchmark calibration I use the upper end of the proposed range, $\tau = 1$. In the last two columns of Tables 2-3, I show what the calibration results look like $\tau = 0.316$, the geometric mean of the $\tau \in \{0.1, 1\}$ suggested by Christiano et al. (2010).

The only parameter estimate that changes substantially is θ, which is lower now. Compared to the benchmark model, output is now more sensitive to changes in inflation (in line with the back-of-the-envelope calculation above), but less sensitive to changes in the bond supply.

Robustness with respect to lower χ^B and lower τ

Again, the only parameter estimate that changes substantially from the benchmark is θ. The comparative statics are more or less equivalent to combining the effects of lower χ^B and lower τ.