AN OPEN-ECONOMY MODEL WITH MONEY, ENDOGENOUS SEARCH AND HETEROGENEOUS FIRMS

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Abstract: This paper describes a new monetary open-economy model where firms have market power due to search frictions in the goods market, and endogenous search effort by consumers mitigates this market power. The optimal inflation rate generally depends positively on the cost of search effort, the cost of firm entry, and the cost of trade. Higher inflation always improves a country's terms-of-trade against its trading partners. I also characterize a general class of matching processes which offer a novel approach to modeling firm sales.

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1 Introduction

Models of price dispersion and the endogenous choice of search effort have found application to many macroeconomic questions, but have been limited to closed economies, and to environments with identical firms. However, allowing firm heterogeneity is important for several reasons. For one, the assumption of identical costs for all firms is prohibitive in a *monetary* model of an open economy, because monetary policy will affect the cost of production, and differently so in each country. For another, if price dispersion is purely due to a friction (such as search costs or sticky prices), it will indicate a distortion of economic equilibrium; but if firms are heterogeneous, price dispersion is an important signal directing consumers to the most efficient producers. In such an environment, if some friction causes all firms to charge the same (high) price, production is inefficient and will result in depressed TFP. The details of market structure and the structure of production, and how monetary policy can affect both, therefore matter for welfare and we need to account for them explicitly.

As the goal is to study such a model of a monetary open economy with endogenous search frictions in the goods market, I proceed in this paper by extending the closed-economy optimal inflation model of Head and Kumar (2005) in four directions. First, I characterize a general class of matching processes that nests all of those used in the literature on endogenous price dispersion building on Burdett and Judd (1983). This class of matching processes is tractable (the number of customers per firm follows a bounded Pareto distribution with shape parameter *less than* or equal to one) and lends itself to empirical applications. Second, I consider firms explicitly which allows me to model free entry in a way analogous to modern trade theory. Third, I solve the model with arbitrary distributions of costs (ex-ante firm heterogeneity), which also facilitates the direct comparison with modern models of international trade. Fourth, I extend the model in a natural way to allow trade, and I characterize a special case which admits simple numerical solutions for an arbitrary number of countries.

In the model, search frictions in the goods market imply that consumers only observe a small subset of all prices, which gives firms market power. In contrast to monopolistic competition models, this market power is not exogenous or only dependent on the prices of other firms, but it also depends on the search effort choices by households. Because this price pressure is an externality, consumers will typically search too little unless prodded to search more through some other friction: in this instance, the inflation tax. As a result, the optimal inflation rate is above the Friedman rule, and is increasing in the cost of search effort. Because search effort and firm entry are complements in generating matches between consumers and firms, the optimal inflation rate is also increasing in the cost of firm entry.

1.1 Related literature

The debt to the closest predecessor of my model, Head and Kumar (2005), has already been acknowledged. Their model applies the price-posting mechanism of Burdett and Judd (1983) to the monetary search economy of Shi (1997), and studies the endogenous choice of search effort in steady state. Head, Kumar, and Lapham (2010) extend the model to study aggregate nominal and real shocks.

There is a small but growing literature that has applied settings with search frictions to questions in international economics. Papers which study the endogenous determination of real exchange rates in a setting with search frictions include Head and Shi (2003), Geromichalos and Simonovska (2014), and Geromichalos and Jung (2015). Shi (2006) discusses currency unions in a monetary search setting, and Devereux and Shi (2013) and Zhang (2014) study the emergence and stability of international reserve currencies.

Modeling a search effort decision by buyers, or an entry decision by buyers or sellers, is almost as old as search theory itself. In addition to Head and Kumar (2005) itself, the following papers are particularly relevant. In an early contribution, Li (1995) studies the "hot potato effect": because buyers do not take the external effects of their search effort into account, a tax on money holdings which increases the rate of matching in general equilibrium can enhance welfare.[1] Liu, Wang, and Wright (2011) also study the hot potato effect. In contrast to this paper, they find that the intrinsic margin of search effort decreases with inflation, but like this paper, they suggest that the optimal rate of inflation can reflect a trade-off between minimizing the inflation tax (which pushes optimal inflation towards the Friedman rule) and mitigating the externality of endogenous search effort (which pushes optimal inflation higher). Wang (2014) develops a model with endogenous search effort and price posting in order to quantify the inflation tax.

Such an externality due to endogenous search effort (or entry into a market) is also studied by Shi (2006), Berentsen, Rocheteau, and Shi (2007), and Craig and Rocheteau (2008). In those papers, moderate inflation increases search effort and can mitigate the externality. Generally, the externality of endogenous search could be positive or negative; this arises from the bargaining mechanisms that determine the terms of trade in their models. In this paper, by contrast, the terms of trade are determined by price posting, and the externality of search effort is always positive, i.e. more search effort puts pressure on firms to reduce markups, but individual buyers do not take this into account.

Recently, some papers have tested empirical implications of the Head and Kumar (2005) model. Caglayan, Filiztekin, and Rauh (2008) find that price dispersion is V-shaped as a function of inflation and that the amount of dispersion is related to search costs. Using European Union price data, Becker and Nautz (2010) and Becker (2011) find that this V-shaped relationship disappears in highly integrated markets, where search costs are presumably low.

The rest of the paper is organized as follows. Section 2 describes the model environment and the decision problems facing households and firms. Section 3 solves the model in the closed economy, and Section 4 describes and solves the model for the open

[1] In the present paper, the extensive margin of matching, the physical velocity of money, is fixed. But if anything, this makes my results stronger, as there is a hot potato effect on the intensive margin of matching alone.

economy. Section 5 concludes and is followed by two appendices: Appendix A contains proofs and detailed derivations of statements in the main test, and Appendix B derives a tractable class of matching processes.

2 Setup of the model

The model extends the optimal inflation framework introduced by Head and Kumar (2005). That framework was based on a monetary search economy (Shi, 1997) with price posting (Burdett and Judd, 1983) and endogenous search effort by buyers.

Time is discrete and infinite. There is a measure 1 of households, whose members can be either shoppers or workers; there is a fixed positive measure of each type, and they sum to measure 1.[2] There exists a government that supplies a stock M_t of perfectly durable, infinitely divisible asset called money at time t, and augments this stock at the beginning of each period with lump-sum transfers T_t to the households. Household members do not have independent utility but share equally in the utility of the household (Shi, 1997). Households value the streams of consumption c, search effort s, and work effort n according to the separable utility function:

$$U(c, s, n) = (1 - \beta) \sum_{t=0}^{\infty} \beta^t \left[u(c) - \mu s_t - n_t\right] \tag{1}$$

Assume $\beta \in (0, 1)$ and $\mu > 0$, with the disutility of labor normalized to 1. $u(\,\cdot\,)$ is strictly increasing, strictly concave, and satisfies the Inada conditions. Furthermore, $u'(c)c$ is non-increasing in c.

Next, assume that there exists a measure N_t of firms, owned by households, that hire workers in a perfectly competitive labor market and produce goods. In order to trade these goods, firms and shoppers enter an anonymous and memoryless goods market

[2] In this model, unlike in some others, variations in search effort or labor effort take place at the intensive margin. With the given preferences and market structure, the extensive margin would not matter even if it was flexible.

characterized by search frictions. The infinitely many shoppers of a household must search separately and, once in the market, cannot coordinate with their siblings.

The trading period proceeds as follows: Firms learn their costs, as well as all macroeconomic variables, and post prices. As they can perfectly forecast demand (by the law of large numbers applied to the mass of shoppers), they then hire workers and produce. Households learn the price distribution and decide how much money m_t each member carries into the market, and how much effort s should be spent on obtaining price offers. Once in the market, each shopper receives k random quotes (Burdett and Judd, 1983), i.e. is able to observe the prices of k firms, where k is drawn from a distribution with cumulative distribution function $Q(k|\eta)$, mean number of quotes $\eta > 0$, and support $\{1, \ldots, K\}$. The parameter η is monotonically increasing in search effort s, and a distribution of quotes with higher search effort s' strictly first order stochastically dominates one with lower search effort $s < s'$: informally, more effort supplies more quotes. The precise set of assumptions sufficient for the results of this paper is stated in appendix B.

Shoppers can then purchase goods from any of the firms whose prices they have observed, but cannot spend more money than they carry. To keep the analysis tractable, shoppers cannot coordinate with one another and cannot recall any quotes from a previous period. As in every paper using the framework of Burdett and Judd (1983), the number of quotes is determined before the shopper learns what these quotes are; the shopper cannot decide to search harder if the quotes are bad. Shoppers remain anonymous to firms, so firms will only accept payment in cash. At the end of the period, firms pay their workers and remit the profits to their owners. Workers take home their pay, shoppers take home their goods, and the members of the household share equally in consumption and earnings.

To simplify notation, all monetary variables will be expressed in "constant-money" terms, which means the nominal value divided by the current money stock M_t. (Head and Kumar (2005) use the term "real", but the international macroeconomics literature understands “real” to mean a nominal value divided by the price level P_t, instead of the money stock.) Deflating nominal variables by dividing them by the money stock is required to keep the interesting variables stationary.

2.1 Matching

Let $q_k(\eta),\ k = 1, \ldots, K$ be the probability mass function of $Q(k|\eta)$, i.e. the probability that a shopper who searches with success parameter η observes exactly k prices. Assume that $\eta = s \cdot N$, where s is search intensity and N is the mass of active firms.[3] Define the function $J :\ [0, 1] \times [0, \infty) \to [0, 1]$:

$$J(F, sN) = \sum_{k=1}^{K} q_k(sN)\,(1 - (1 - F)^k)$$

Let the cumulative distribution function of prices posted by sellers be $F_t(p_t)$ on support $\mathcal{F}_t$. Naturally, shoppers who observe more than one price will only buy from the cheapest firm. Then the c.d.f. of the transactions prices is the c.d.f. of the lowest price observed by a buyer, which happens to be:

$$J(F_t(p_t), s_t N_t) \quad \forall p_t \in \mathcal{F}_t.$$

Similarly, how many transactions can a firm with price p_t expect when all shoppers search with success rate sN? If the price distribution has no mass points, then neither does the distribution of transactions, and the answer is given by the derivative of J with respect to its first argument, denoted by $a :\ [0, 1] \times [0, \infty) \to [0, \infty)$:

$$\begin{aligned} J_1(F_t(p_t), s_t N_t) &= a(F_t(p_t), s_t N_t) \quad \forall p_t \in \mathcal{F}_t \\ a(F, sN) &= \sum_{k=1}^{K} q_k(sN)\, k\, (1 - F)^{k-1} \end{aligned} \tag{2}$$

[3] Matching success will depend both on shopper effort and on firm presence. Through this channel, firm entry will put downward pressure on prices.

For this reason, I call $a(F, sN)$ the "arrival function" of successful matches in analogy to the "arrival rates" common in search models, but because consumers have constant expenditure here, it can also be interpreted as describing sales per firm relative to the average. (Note that $a(1) < 1 < a(0)$.) Appendix B derives closed form solutions for J and a for a general class of matching processes that is highly tractable and nests all of those used in the literature.

2.2 Households

Households take as given the distribution of prices $F_t(p_t)$ and the wage w_t (all expressed in constant-money terms), and choose the search effort and expenditure strategies for its shoppers, as well as the work effort of its workers. As the sub-utility of consumption $u(c)$ is strictly concave, the household will treat all of its shoppers the same. Therefore, all shoppers pursue the same expenditure strategy $x_t(p_t)$ at time t. With household money stock m and aggregate money stock M, the household optimizes:

$$v(m, M) = \max_{m', x(\cdot), c, s, n} \{u(c) - \mu s - n + \beta \mathbb{E} \{v(m', M') | M\}\} \tag{3}$$

subject to:

$$\frac{m'}{m} = 1 - \int_{\mathcal{F}} \frac{x(p)}{m} a(F(p), sN)\ dp + \Pi(M) + wn + \frac{T}{m} \tag{4}$$

$$c = \int_{\mathcal{F}} \frac{x(p)}{p} a(F(p), sN)\ dp, \tag{5}$$

where $\Pi(\cdot)$ denotes aggregate profits. We can interpret the budget constraint (4) as follows: a household's money holdings would be constant ($m'/m = 1$) if not for total expenditure on consumption goods (an aggregate of per-meeting expenditures $x(p)$), profit and wage income, and the monetary transfer. The level of aggregate consumption is derived by measuring the purchase of each shopper as expenditure divided by the price, then adding up over all shoppers.

Denote the Lagrange multiplier on the budget constraint (4) by Ω_t.[4] A shopper carrying m_t units of currency cannot buy more than $\frac{m_t}{M_t p_t}$ of goods at nominal price $M_t p_t$ (p_t in constant-money terms). Because all buyers share equally in their household's utility and are individually too small to influence consumption c_t, they will pursue a reservation price strategy, such that:

$$x_t(p) = \begin{cases} m_t & \text{if } p \leq \bar{p}_t \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

The key constraint is that once in the market, they cannot transfer the money to another shopper. Going home means leaving the money idle until next period; and the value of idle money at the end of period t is $M_t \Omega_t$, hence the nominal reservation price satisfies:

$$M_t \, \bar{p}_t = M_t \, \frac{u'(c_t)}{\Omega_t} \tag{7}$$

The labor supply is perfectly elastic: households will meet any demand for labor as long as the wage satisfies $w_t \geq 1/\Omega_t$.

Let $c(s)$ denote a household's consumption resulting from having all shoppers search with effort s. By the law of large numbers, this is a deterministic function. If it is also concave (which is to be verified), then the optimal choice of search effort implies that $u'(c(s))c'(s) = \mu$.

Finally, the key intertemporal variable, Ω, is determined by the first-order condition for m_{t+1} together with the envelope condition for m_t, iterated by one period:

$$\frac{\Omega_t}{m_t} = \beta \, \mathbb{E}_t \left\{ u'(c_{t+1}) \frac{c_{t+1}}{m_{t+1}} \right\}. \tag{8}$$

[4] Ω_t will be stationary because the budget constraint is expressed in constant-money terms.

2.3 Firms

In this section, for the purpose of readability, I suppress time subscripts and the matching rate argument, and write the arrival function $a(F, sN)$ as simply $a(F)$. The total measure of all firms is $N > 0$.

Assume that a firm requires ϕ units of labor to produce one unit of output. In a perfectly competitive labor market, firms can hire any quantity of labor at the nominal wage Mw. With nominal price Mp, they will attract $a(F(p))$ shoppers (the price distribution $F(p)$ is expressed in constant money terms so as to be stationary). If $p \leq \bar{p}$, each shopper will spend all the money they carry, and as shown above, all shoppers carry the same amount of money, M. Therefore, the nominal profits of a firm with marginal cost ϕ are determined by:

$$M\pi(\phi) = \max_p \left\{ M \left(1 - \frac{\phi w}{p}\right) \frac{a(F(p))}{N} \;\middle|\; p \leq \bar{p} \right\}. \tag{9}$$

As shown in Head and Kumar (2005), when all firms have the same marginal cost ϕ, they must all make the same profits; in particular, the same profits as any firm which charges the reservation price. This allows us to solve

$$F(p) = a^{-1}\left(a(1)\frac{1 - \frac{\phi w}{\bar{p}}}{1 - \frac{\phi w}{p}}\right), \tag{10}$$

for the price distribution $F(p)$ if we know $a(\cdot)$.

When firms have heterogeneous costs, however, they will not all make the same profits. The firm's problem must be solved in the following way: assume that each firm draws a value of ϕ, the marginal labor cost of production, from a continuous distribution $G(\phi)$ with support $\mathcal{G} \subset \mathbb{R}$ and upper bound $\bar{\phi}$. (The nominal marginal cost of production is then ϕMw.) Abusing notation, ϕ will also identify a representative firm with cost ϕ.

Lemma 2.1. *Assume p_1 and p_2 are solutions to profit maximization with ϕ_1 and ϕ_2, respectively* *(we do not require them to be unique solutions). If $\phi_1 < \phi_2$, then $p_1 < p_2$.*

Proof. See appendix A. □

Consequently, firms' prices are completely ranked by their costs. If the support of G includes $\bar{p}/w$, i.e. $\bar{\phi}w > \bar{p}$, some firms (with mass $(1-G(\bar{p}/w))$) will not be able to sell their products for any profit at all. In particular, if $\bar{\phi} = \infty$ then $G(\bar{p}/w)) < 1$ always. To simplify matters, I assume that all such noncompetitive firms are simply inactive, and are ready to produce as soon as the reservation price rises. Define the cost distribution conditional on producing:

$$\tilde{G}(\phi) = \frac{G(\phi)}{G(\bar{p}/w)} \tag{11}$$

If G is differentiable, then the solution to profit maximization is a differentiable function $p(\phi)$, and the price distribution $F(p)$ is also differentiable. We get the following ranking conditions:

$$F(p(\phi)) = \tilde{G}(\phi) \text{ for } \phi w \leq \bar{p} \tag{12}$$

$$F'(p(\phi))p'(\phi) = \tilde{G}'(\phi) \tag{13}$$

Profit maximization implies the following first-order condition:

$$\frac{\partial \pi}{\partial p}(p;\phi) \ = \ \left(1 - \frac{\phi w}{p}\right) \frac{a_1(F(p))}{N} F'(p) + \frac{\phi w}{p^2} \frac{a(F(p))}{N} \ = \ 0 \tag{14}$$

Together, (12), (13), and (14) determine a differential equation

$$p'(\phi) = \left(\frac{p(\phi)^2}{\phi w} - p(\phi)\right) \left(\frac{-a_1(\tilde{G}(\phi))}{a(\tilde{G}(\phi))}\right) \tilde{G}'(\phi) \tag{15}$$

with boundary condition

$$p(\bar{p}/w) = \bar{p}. \tag{16}$$

Equation (15) is a special case of the Riccati differential equation, which has the following solution.

Lemma 2.2. *Let $p(\phi) = \infty$ if $\phi w > \bar{p}$ and*

$$p(\phi) = \frac{a(\tilde{G}(\phi))\bar{p}}{a(1) - \frac{\bar{p}}{w}\int_{\phi}^{\bar{p}/w} a'(\tilde{G}(t))\frac{1}{t}\tilde{G}'(t)\,dt} \tag{17}$$

otherwise. Then $p(\phi)$ solves the system $\{$(15), (16)$\}$ *and therefore maximizes profits* (9).

Proof. See appendix A. □

3 Equilibrium in a closed economy

Even without free entry, the mass of active firms can potentially vary with the reservation price. Denote the number of existing firms by $\bar{N}$, and the mass of active firms by $N = \bar{N}G(\bar{p}/w)$ in the case where the firms have different marginal costs. Define the gross money growth rate $\gamma_t = 1 + \frac{T_t}{M_t}$, and assume that the stochastic process of T_t (and therefore M_t) is such that γ_t is stationary.

Definition 1. A *stationary monetary search equilibrium* (SMSE) is a collection of a value function $v(m, M)$, policy functions $m'(m, M)$, $x(\,\cdot\,; m, M)$, $s(m, M)$, $n(m, M)$, common expenditure rule, $X(\,\cdot\,; M)$, common search effort $S(M)$, a wage $w(M)$, a distribution of posted prices, $F(\,\cdot\,; M)$, and expectations $\gamma'(M)$ such that

1. The value function $v(m, M)$ solves (3) with the associated policy functions $m'(m, M)$, $x(\,\cdot\,; m, M)$, $s(m, M)$, and $n(m, M)$, and Lagrange multiplier $\Omega(M)$.

2. The price distribution $F(\,\cdot\,; M)$ satisfies (10) in case all firms have the same marginal cost, or (12) together with (17) otherwise.

3. Aggregate (constant money) profits in (4) are $\Pi(M) = \left(1 - \frac{\phi w(M)}{\bar{p}(M)}\right) a(1, SN)$ in case all firms have the same marginal cost, or

$$\Pi(M) = \int_{\mathcal{F}} \left(1 - \frac{\phi w(M))}{p}\right) a(F(p; M), S(M)N)\; dp$$

if they do not.

4. The money market clears: $w(M)n(M, M) + \Pi(M) = 1$. By Walras' Law, this is equivalent to labor market clearing.

5. Individual choices equal aggregate quantities: $x(p; M, M) = X(p; M)$ for all p, $s(M, M) = S(M)$, and $m'(M, M) = \gamma M$.

6. Expectations $\gamma' \,|M$ are rational.

7. Money has value: $F(p) < 1$ for some $p < \infty$.

Additional uncertainty about parameters would be easy to incorporate, as Ω_t fully summarizes all intertemporal expectations and all utility and cost parameters only affect the equilibrium contemporaneously. In the aggregate, we can write:

$$\Omega_t = \mathbb{E}_t \left\{ \frac{\beta}{\gamma_{t+1}} u'(c_{t+1}) c_{t+1} \right\}. \tag{18}$$

As the labor market is perfectly competitive and the marginal disutility of working is constant, the wage must exactly compensate workers for their effort:

$$w = \frac{1}{\Omega}. \tag{19}$$

3.1 Equilibrium conditional on fixed search effort

Equation (5) can be combined with the optimal pricing formula to express aggregate consumption as a function of aggregate search effort s (more details in appendix A.1). Consider first the case where all firms have the same marginal cost ϕ, which provides valuable intuition:

$$c(s) = \frac{1 - a(1, sN)}{\phi w} + \frac{a(1, sN)}{\bar{p}}, \tag{20}$$

Note that $a(1, \cdot)$ equals $q_1(\cdot)$, i.e. the fraction of consumers who receive a single quote. In this sense, the outcome is equivalent to a market in which prices are subject to Bertrand-style negotiation. If shoppers who had more than one quote were able to bring the price down to the efficient level ϕw, while all those that didn't had to accept their reservation price $\bar{p}$, the market outcome would be the same as here. However, this intuition does not carry through to the optimal choice of search effort: the function $c(s)$ is derived for a symmetric equilibrium where all shoppers choose the same amount of search effort. As no single shopper can affect the search effort of their fellow shoppers, they will generally not choose a socially optimal amount of effort.

Substituting (7) and (19), and rearranging, we derive the *market equilibrium* equation:

$$u'(c)c = \Omega \left(\frac{u'(c)}{\phi}(1 - a(1, sN)) + a(1, sN) \right) \tag{21}$$

In order to obtain a relationship in (s, c)-space, we have to replace Ω. There are three meaningful ways of doing this.

Constant expectations: When expectations of future consumption and money growth are fixed (e.g. if we are concerned with the effects of a purely temporary shock), the curve in (s, c)-space is downward sloping for low Ω, becomes close to vertical for intermediate values of Ω, and then upward sloping for high Ω. It shifts rightward with increasing Ω (higher s for any given c).

Steady state: Assume that there is no uncertainty and $c_{t+1} = c_t$. Then equation (21) becomes:

$$\frac{\gamma}{\beta} = \frac{u'(c)}{\phi}\left(1 - a(1, sN)\right) + a(1, sN). \tag{22}$$

This curve is upward sloping in (s, c)-space, and shifts right/down for higher money growth γ. Therefore, higher money growth implies lower consumption for a fixed level of search effort. Consequently, as long as search effort is fixed above zero, the optimal rate of steady-state money growth satisfies $\gamma = \beta$ (the "Friedman rule"). The right-hand side achieves its minimum of 1, which implies the first-best $u'(c) = \phi$.

Nominal interest rates: It is easy to introduce nominal risk-free bonds into the household's problem (4). The nominal interest rate can be found assuming zero net supply of these bonds:

$$\begin{aligned} 1 + i_t &= \frac{u'(c_t)c_t}{\Omega_t} \qquad (23) \\ &= \left[\mathbb{E}_t\left\{\frac{\beta}{\gamma_{t+1}} \times \frac{u'(c_{t+1})c_{t+1}}{u'(c_t)c_t}\right\}\right]^{-1} \end{aligned}$$

Equation (21) can then be written as:

$$i = \left(\frac{u'(c)}{\phi} - 1\right)\left(1 - a(1, sN)\right). \tag{24}$$

This expression does not presume a steady state or even certainty, because the nominal interest rate summarizes all expectations about the future. It demonstrates that high nominal interest rates cause low consumption (strong market power) and/or higher search (shoe-leather costs). Any policy that achieves $i = 0$ implements the first-best $u'(c) = \phi$, as long as sN is fixed above zero.

Consider now the case where firms have heterogeneous marginal costs. Denote the infimum of the support of $G(\cdot)$ by $\underline{\phi}$. The upper bound is given by the reservation cost $\bar{p}/w$, which we can now see to equal $u'(c)$. The number of active firms is therefore $N(c) = N\,G(u'(c))$. The market equilibrium equation becomes:

$$u'(c)c = \Omega \left[a(1, sN(c)) + \int_{\underline{\phi}}^{u'(c)} \frac{u'(c)}{t} \Big(-a_1(\tilde{G}(t), sN(c)) \Big) \tilde{G}(t)\tilde{G}'(t)\ dt \right] \tag{25}$$

This curve is increasing in (s, c)-space, similar to the simpler case above. The endogenous reservation price makes things more complicated, because not all firms may be able to compete. However, this complication is not very severe, especially when the cost distribution $G(\phi)$ is well-behaved and does not put too much weight on the tails. In steady-state terms:

$$\frac{\gamma}{\beta} = a(1, sN(c)) + \int_{\underline{\phi}}^{u'(c)} \frac{u'(c)}{t} \Big(-a_1(\tilde{G}(t), sN(c)) \Big) \tilde{G}(t)\tilde{G}'(t)\ dt \tag{26}$$

Outside of a steady state, the nominal interest rate analogue of (24) is obtained by setting $1 + i$ equal to the right-hand side of (26).

3.2 Endogenous search

As discussed in the previous section, the socially optimal monetary policy is the Friedman rule when search effort held constant, because the Friedman rule minimizes firms' market power. This result still holds when search effort is socially chosen, by maximizing $u(c) - \mu s$ subject to the constraint (21) or (25). However, it breaks down when households privately choose their search effort. The reason is that search effort accomplishes two things: it allows shoppers to pick up low-price offers, and through that, it constrains the market power of the firms. As each household is small, only the former effect is internalized, and as a consequence, loose monetary policy creates price dispersion and forces shoppers to search harder.

Solving the households' problem (equation 3), privately optimal search implies that

$u'(c(s))c'(s) = \mu$, taking the distribution of prices $F(p)$ as given. Details of the derivation are in appendix A.2; we need to define the auxiliary function

$$h(F, \eta) = \int_0^F \frac{a_2(z, \eta)}{a(z, \eta)} \, dz \tag{27}$$

(the index a_2 refers to the derivative with respect to the second argument), and can then derive the *optimal search* equation for the case where all firms have the same cost:

$$\mu = \Omega \, N \left[-a(1, sN)h(1, sN) \right] \left(\frac{u'(c)}{\phi} - 1 \right) \tag{28}$$

We can use the market equilibrium equation (21) to divide out Ω and obtain a purely contemporaneous relationship:

$$\mu = N \, u'(c)c \, \frac{-a(1, sN)h(1, sN) \left(\frac{u'(c)}{\phi} - 1 \right)}{\frac{u'(c)}{\phi}(1 - a(1, sN)) + a(1, sN)} \tag{29}$$

The behavior of the denominator and of the markup $(u'(c)/\phi - 1)$ is monotonic. If furthermore $u'(c)c$ is non-increasing in c,[5] the behavior of equation (29) in (s, c)-space is driven by the term $-a(1, sN)h(1, sN)$.

Recall first that $a(1, \cdot)$ equals $q_1(\cdot)$, i.e. the fraction of consumers who receive a single quote. This fraction is a strictly decreasing function of search effort, and it seems reasonable to assume that it approaches zero as search effort becomes infinite. Also note that $h(1, s) < 0$ for $s > 0$ and $h(1, s) = 0$ as $s = 0$, and it is continuous in s (proofs in appendix A), so it must decrease for low levels of search effort. Consequently, the term $-a(1, sN)h(1, sN)$ is positive for $s > 0$ and zero for $s = 0$; it must therefore strictly increase for low levels of search effort, and then decline back to zero for high levels of search effort (lemma A.1). As a result, equation (29) describes a hump-shaped curve in (s, c)-space.

[5] Head and Kumar (2005) make this assumption throughout. If $u'(c)c$ was increasing in c, there could be multiple equilibria, or none.

This curve is purely contemporaneous, unaffected by expectations.

Going back, we can state:

Theorem 3.1. *If the following conditions hold then an SMSE* $\{(\Omega_t, c_t, s_t)\}_{t=0}^{\infty}$ *exists, is unique, and is fully described by* (18)*, and by* (21) *and* (29) *for each time period* t*.*

E1. The matching process satisfies assumptions M1–6 in appendix B.

E2. The sub-utility of consumption is such that $u'(c)c$ *is strictly decreasing in* c*.*

E3. The money growth process γ_t *is stationary and such that* $\Omega < u'(c)c$ *in all states and all time periods.*

Proof. See Appendix A. □

For suitable matching processes (see appendix B), this optimal choice of search effort is a robust interior equilibrium.[6]

Consider again the case where firms have different marginal costs. Recall that $\underline{\phi}$ is the infimum of the support of $G(\cdot)$, that the firms with a cost higher than $\bar{p}/w = u'(c)$ do not produce, and that the number of active firms $N(c) = \bar{N}G(u'(c))$ is therefore declining in c. The optimal search equation becomes:

$$\mu = \Omega\, N(c) \left[a(1, sN(c))h(1, sN(c)) \; + \int_{\underline{\phi}}^{u'(c)} \frac{u'(c)}{t} \left(-a_1(\tilde{G}(t), sN(c))\right) h(\tilde{G}(t), sN(c))\, \tilde{G}'(t)\; dt \right] \tag{30}$$

The optimal search equation has the same shape in (s, c)-space as in the simpler case of identical firms, after substituting out Ω using (25). Under the same sufficient conditions, with two additions, an SMSE $\{(\Omega_t, c_t, s_t)\}_{t=0}^{\infty}$ exists, is unique, and is fully described by

[6] This was not the case in the original model of Head and Kumar (2005). The point is that consumption as a function of search effort, taking prices as given, is strictly concave only for certain matching processes, but convex for others. In the latter case, households must pursue mixed strategies in any symmetric equilibrium.

(18), (25), and (30) for each time period t. The additional conditions require that the cost distribution $G(\phi)$ is well-behaved: (1) it does not put too much weight on the tails, and (2) it does not have multiple modes. In the latter case, existence survives, but there may be multiple equilibria.

The equilibrium conditions of the minimal model (with autarky and a fixed measure of homogeneous firms) are illustrated in Figure 1.

3.3 Free entry

Consider the case when all firms have the same cost. Say that entry is subject to a fee $\kappa < 1$, assessed in constant-money terms and each period.[7] Assume that this entry fee is distributed among households, i.e. added to equation (4). Firms will enter if their profits exceed the entry fee, and exit otherwise. This yields the free-entry condition:

$$\begin{aligned} \kappa N &= \Pi \\ &= \left(1 - \frac{\phi}{u'(c)}\right) a(1, sN) \end{aligned} \tag{31}$$

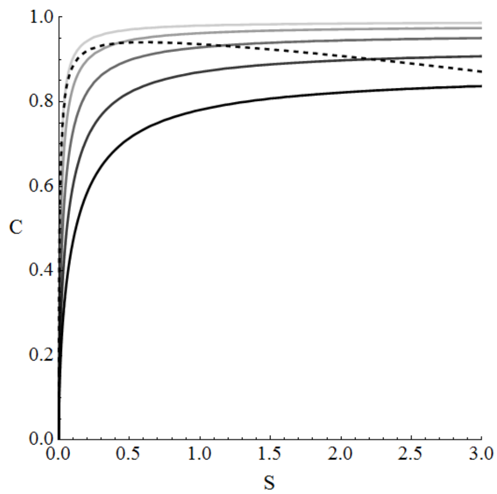
The SMSE is now expressed as $\{(\Omega_t, c_t, s_t, N_t)\}_{t=0}^{\infty}$, which solve equations (18), (21), (29), and (31) for each t. Dividing equation (29) by equation (31) yields a curve in (sN, c)-space that takes the role of equation (29) in the previous discussion. The market equilibrium equation (21) depends only on sN, not on s or N directly. The only difference is that the new optimal search curve is a bit flatter than previously, but it has the same shape, and the SMSE exists and is unique under the same conditions as before.

A similar analysis can be conducted in the case where firms have heterogeneous costs. Define the function $\Gamma(\phi) = \int_{\underline{\phi}}^{\phi} t\,\tilde{G}'(t)\,dt$ (the conditional mean below ϕ). Aggregate profits are, in constant-money terms (derivation in Appendix A.5):

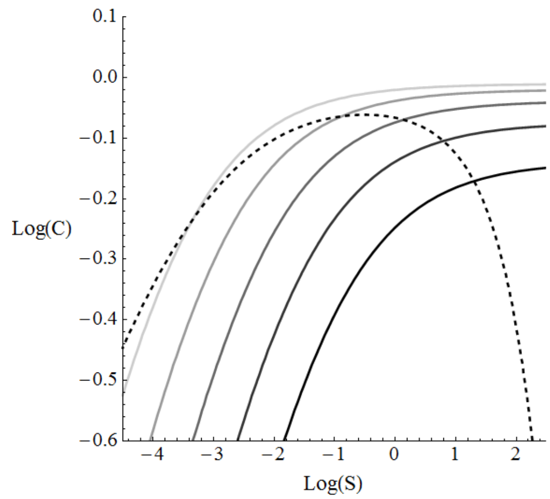
[7] The qualitative results are the same if the cost is assessed in labor terms.

Figure 1: **General equilibrium in the minimal model: autarky, homogeneous firms, no entry.** The parameters are $\phi = 1$, $\beta = 0.98$, $\sigma = 2$, and $\mu = 0.008$. The continuous lines represent market equilibrium, and the contours are such that steady-state nominal interest rates would be $\{2\%, 4\%, 8\%, 16\%, 36\%\}$ (left to right, light to dark). The dashed lines represent the contemporaneous optimal search equation.

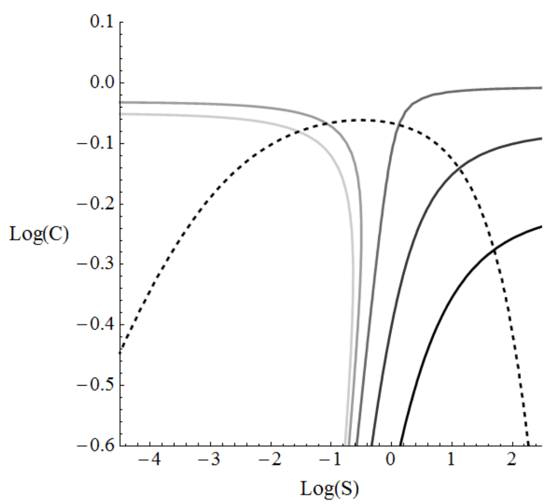
(a) Continuous: equation (22) (steady state). Dashed: equation (29).



(b) As panel (a), but in logs instead of in levels.



(c) As panel (b), except that future consumption is expected to be $e^{-0.07}$.



$$\Pi = 1 - a(1, sN(c))\frac{\Gamma(u'(c))}{u'(c)} - \int_{\underline{\phi}}^{u'(c)} \frac{\Gamma(t)}{t} \left(-a_1(\tilde{G}(t), sN(c))\right) \tilde{G}'(t)\, dt \tag{32}$$

The rest follows from $N(c) = \bar{N}G(u'(c))$, setting $\Pi = N(c)\kappa$, and making $\bar{N}$ an endogenous variable. Again, the nature of equilibrium does not change materially.

3.4 Comparative statics

For details and proofs relating to this section, see Appendix A.4.

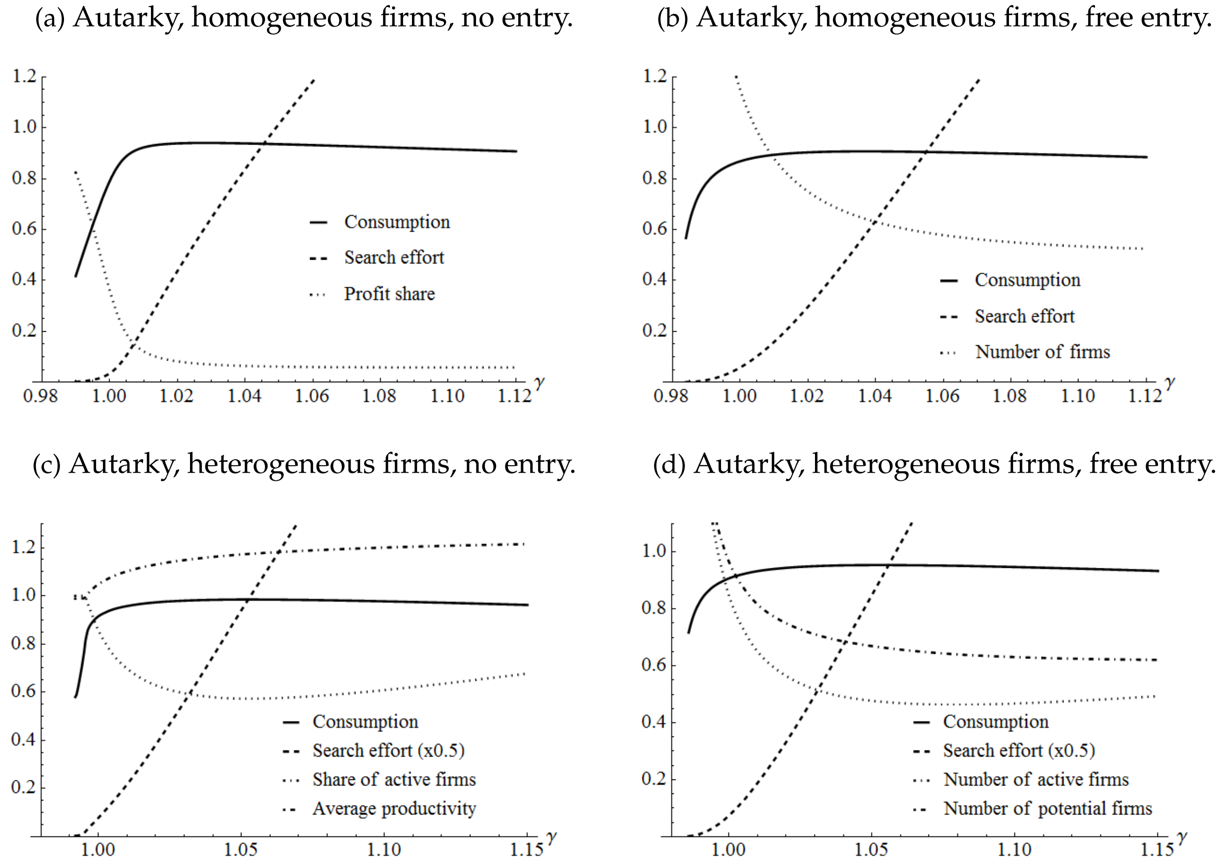
What are the effects of real shocks on the steady-state equilibrium? The effect of an increase in the production cost ϕ is a fall in consumption and a rise in search effort. On the other hand, an increase in the disutility of search, μ, leads to a fall in both consumption and search effort. The same result holds when firms have heterogeneous costs, if we define a cost shock in the first-order stochastic dominance sense: If $G_2(\phi) < G_1(\phi)$ for all ϕ, then G_2 implies lower consumption and higher search effort than G_1 in steady state. Thirdly, an increase in the cost of firm entry lowers consumption, and generally, but not necessarily, reduces search effort.

Only expectations of future money growth $\{\gamma_t\}$ affect the value of money, Ω, and therefore the real economy; surprise money growth shocks that do not affect expectations have no real effect in this model (prices are fully flexible and money is neutral). Therefore, I focus on steady states with constant and expected money growth γ each period. As shown in Appendix A.4, there exists a consumption-maximizing rate of money growth $\hat{\gamma} > \beta$. Consumption is increasing in γ for $\beta < \gamma < \hat{\gamma}$ and decreasing in γ otherwise. Search effort is increasing in γ throughout. With free entry, the number of firms is decreasing in γ. These results are illustrated in Figure 2.

If we define the aggregate nominal price level by

$$P_t = \frac{M_t}{c_t}, \tag{33}$$

Figure 2: **Effects of steady-state money growth.** The parameters are $\beta = 0.98$, $\sigma = 2$, and $\mu = 0.008$ for all panels; $\phi = 1$ for panels (a) and (b); $G(\phi) \sim \log\mathrm{N}(1, \frac{1}{6})$ for panels (c) and (d); $\kappa = 0.2$ for panels (b) and (d).



For all illustrations, the representative utility function is:

$$U\left(\{s_t, c_t\}_{t=0}^{\infty}\right) = (1-\beta)\sum_{t=0}^{\infty}\beta^t\left[\frac{c_t^{1-\sigma}-1}{1-\sigma} - \mu s_t - L(s_t, c_t)\right].$$

For the matching process in Figures 1 and 2, k follows a **geometric** distribution ($\rho = 1$) conditional on $k > 0$: $q_k(\eta) = \frac{\eta^{k-1}}{(1+\eta)^k}$, which yields $\bar{k} = 1 + \eta$ and $a(F, \eta) = \frac{1+\eta}{(1+\eta F)^2}$. (This process satisfies conditions M1–M6 defined in Appendix B.) Intuitively, this matching process can be interpreted as the shopper flipping a (loaded) coin until it comes up tails, and then receiving one price quote for each flip.

Aggregate work effort $L(s_t, c_t)$ is defined in Equations (49) and (50), and the disutility of work effort is normalized to 1.

we can also discuss the effects of money growth on inflation. Clearly, in the long run, inflation must equal money growth (net of consumption growth). In the short run, however, they are separate objects, because of the effect of money on consumption growth. Consider a state where expectations of money growth are so low that consumption would increase with higher money growth, and assume that innovations to money growth have a positive autocorrelation. Then a negative shock to money growth will decrease expectations of further money growth and cause a decline in consumption. Consequently, price inflation will decline by less than money growth and may even rise in rare cases.

Lastly, it is worth paying attention to the interaction between the search cost μ and the optimal rate of money growth in steady state. A higher μ shifts the optimal search curve down in (s, c)-space, implying a lower level of consumption for a given search effort. Assume that γ was previously such that the consumption maximum was achieved. As the market equilibrium curve slopes up for any level of γ, the new consumption maximum is to the right and below the formerly optimal market equilibrium curve. Consequently, the optimal rate of inflation is increasing in the search cost μ; and this is also true for the cost of firm entry in the model with free entry of firms, because the number of firms N and the effort of searchers s are complements in generating matches between consumers and firms.

3.5 Pass-through and price dispersion

What are the distinctive implications of a model of price posting and price dispersion such as this one, compared to traditional approaches in the industrial organization and trade literatures? Consider the monopolistic competition model of Hopenhayn (1992) and Melitz (2003), where products are differentiated by consumer preferences, markups are constant by construction (and therefore the cost pass-through elasticity is 1 for every firm), and price dispersion exactly reflects cost dispersion (so inputs are efficiently allocated). Here, by contrast, the product is homogeneous but producers are differentiated by search frictions, markups are endogenous, price dispersion does not necessarily reflect cost dispersion, and inputs are not efficiently allocated; the most efficient firm 'should' be doing all the production, but the problem is that few consumers know how to find it (reliably).

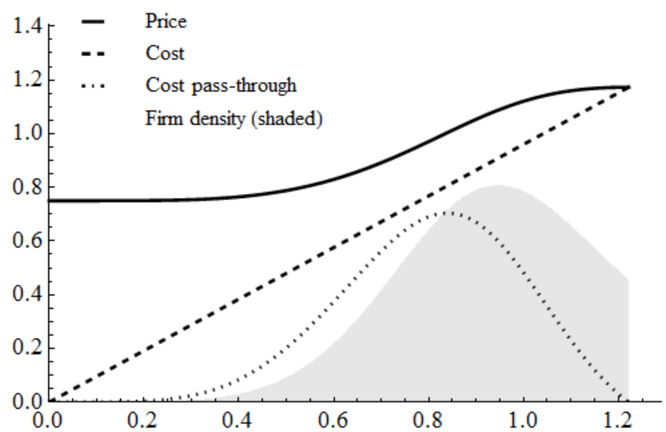
Let us look at markups, pass-through, and price dispersion in more detail. Certainly, there are monopolistic competition models of "endogenous markups" (e.g., Feenstra, 2003, based on translog preferences, and Simonovska, 2015); but in those models, markups are endogenous only to price competition. If consumer search effort is important in disciplining firm pricing, then a richer theory of markups is useful. Furthermore, in Feenstra's model of translog preferences, the least efficient firms respond the most to costs; the pass-through elasticity (of an idiosyncratic cost shock) of the marginal firm is 1, while that of the most efficient firms is lower. In the model of Bernard, Eaton, Jensen, and Kortum (2003) which mixes monopolistic competition among products categories with Bertrand competition within a product category, on the other hand, the most efficient firms have the highest pass-through and the marginal firms have pass-through close to zero on average.[8] As Figure 3 reveals, cost pass-through in my model is quite different. The most and the least efficient firms have zero pass-through, but for different reasons: the most efficient firms charge high prices which are not sensitive to costs, and the least efficient firms always charge the reservation price. The intermediate firms have the highest pass-through elasticities – which can potentially exceed 1 – because their price is determined by the density of direct competitors with similar costs.

The model can also be used to illuminate some facts about price dispersion. Price dispersion of similar goods offered by different sellers, or in different locations, is often seen as a measure of market imperfection. However, if sellers have different costs, then charging different prices is an important signal to consumers, directing them to the most

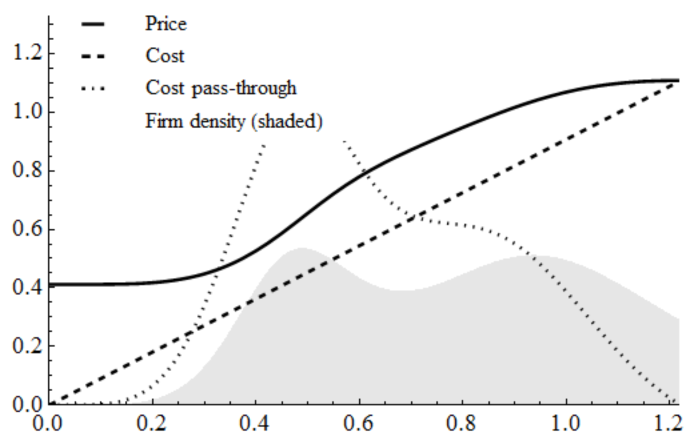
[8] The most efficient firms are most likely to have no direct Bertrand-competitors, therefore their price is a constant markup over marginal cost. The least efficient firms are likely to have a direct competitor, so their price equals their competitor's cost, not their own.

Figure 3: **Pricing and pass-through of firm costs.**
Panels (a) and (c) have a log-normal cost distribution, and panel (b) has a mixture of two such distributions (intended to represent domestic firms and exporters). Pass-through is defined as the elasticity $d\log(p(\phi))/d\log(\phi)$; see Equation (15).

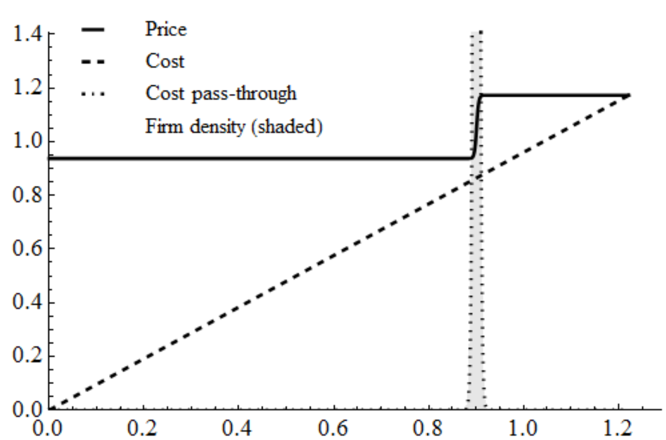
(a) Unimodal cost distribution.



(b) Bimodal cost distribution.



(c) Minimal cost dispersion.



efficient seller. So the question of whether price dispersion is efficient comes down to whether it mostly reflects a signal (cost dispersion) or mostly noise (trading frictions). This is well illustrated by Figure 3, too. In the top two panels, there is significant price dispersion, but it tracks cost dispersion closely (except for a small measure of highly efficient firms). In the bottom panel, by contrast, cost dispersion is almost zero but price dispersion is not; the reason is that price dispersion is bounded from below by the mechanism of the Burdett-Judd model, where price dispersion is the only equilibrium outcome even when all firms have identical costs.

4 The open economy case

4.1 Two countries, with general cost distributions

Let there be two countries, Home and Foreign, and denote Home variables with the subscript H and Foreign variables with the subscript F. Each country has its own currency, and households in a country can only recognize their own currency: they will never accept nor hold foreign currency. Firms which sell goods abroad are therefore paid in the currency of the consumer; however, at the end of the period, they will need to pay their workers and shareholders in domestic currency. In order to exchange currency, firms can access a perfectly competitive foreign exchange market. In that market, we must have balance of payments, and therefore the exchange rate between the two currencies is determinate. Define the nominal exchange rate E in terms of Home currency divided by Foreign currency, and define the real exchange rate using the Home and Foreign price levels: $\varepsilon = \frac{E\,P_F}{P_H}$ (in terms of Home consumption divided by Foreign consumption). By equation (33), this implies:

$$\varepsilon = E\,\frac{M_F\;c_H}{M_H\;c_F} \tag{34}$$

The nominal and real exchange rates are of practical interest, but the model is much easier to solve in terms of the labor cost exchange rate (in terms of Home labor divided by Foreign labor; loosely corresponding to using a PPI instead of the CPI to deflate the nominal exchange rate), because this is the exchange rate relevant for firms' competition in international trade. Define:

$$\begin{aligned} v &= E \, \frac{M_F \, w_F}{M_H \, w_H} \\ &= \varepsilon \, \frac{c_F \; \Omega_H}{c_H \; \Omega_F}. \end{aligned} \tag{35}$$

Trade proceeds as follows. Firms can advertise prices in either country, in local currency. From the point of view of the consumer, domestic and imported goods are indistinguishable. There is also no bias on how likely a consumer is to observe a domestic or importing firm. Whenever a consumer likes the price and wants to purchase the good, the firm produces the good and ships it to they buyer. Of course, goods cannot be shipped for free. Trade costs are assessed in the familiar iceberg form: in order to sell x units of a good in a foreign market, a firm needs to ship τx units from home, with $\tau > 1$. So while there is no bias in preferences against imported goods, domestic firms may be able to charge lower prices on average than the importing competition.

However, solving a version of the model where all firms in a country have the same costs presents technical difficulties; as a result, it makes most sense to proceed with heterogeneous firms from here on.[9] The model with heterogeneous firms has the advantage that the cost distributions of domestic producers and importers will always overlap, and

[9] Even if domestic and foreign competitors happen to have the same labor unit cost in each of their home countries, even without any trade costs, they may have different costs in the market as real wages will vary with the state of the economy. It is not too difficult to solve for market outcomes when the cost distribution has two points of support – domestic and foreign firms – but the real problem is that this equilibrium would come in nine separate regimes: one in which Foreign importers have lower costs than Home producers even accounting for trade costs, one vice versa, one where trade costs make domestic producers cheaper than importers in each country, and two where firms in one country cannot even compete at the other country's reservation price, plus four boundaries on which firms in some country would be indifferent to producing or exporting. This is inconvenient to solve for two countries, and prohibitive for more than two.

as the reservation price varies, the number of active firms in a market varies continuously and never jumps.

In order to simplify the description, I assume that the cost distribution of domestic firms is the same in each country, and that any firm can export if it is able to compete abroad. Assume that the mass of potential firms is N_H at Home and N_F in the Foreign country. The mass of firms actively selling in either country is therefore:

$$N_H(c_H, v) = N_H\, G\Big(u'(c_H)\Big) \;+\; N_F\, G\left(\frac{u'(c_H)}{v\tau}\right) \tag{36}$$

$$N_F(c_F, v) = N_H\, G\left(\frac{u'(c_F)v}{\tau}\right) \;+\; N_F\, G\Big(u'(c_F)\Big) \tag{37}$$

And the distribution of costs among active firms is:

$$\tilde{G}_H(\phi) = \frac{N_H\, G(\phi) + N_F\, G(\frac{\phi}{v\tau})}{N_H(c_H, v)} \tag{38}$$

$$\tilde{G}_F(\phi) = \frac{N_H\, G(\frac{\phi v}{\tau}) + N_F\, G(\phi)}{N_F(c_F, v)} \tag{39}$$

As before, the distribution of quotes a shopper receives depends not only on his search effort, but also on the mass of active firms.

To avoid having to track the bounds of the support of the cost distribution G, I assume that this support is $[0, \infty)$. Existence of an equilibrium requires that the density $G'(\phi)$ declines to zero quickly enough as $\phi \to 0$.

4.1.1 Balance of payments

After trading goods, firms can visit a perfectly competitive foreign exchange market in order to obtain the domestic currency valued by its workers and owners. Without asset markets, or in steady state, currency flows must exactly offset each other. The value of Foreign currency revenue gained by Home exports must equal the value of Home currency revenue paid for Home imports. After substituting equation (35) for the nominal exchange rate, we obtain the balance of payments relationship:

$$v = \frac{\Omega_H}{\Omega_F} \frac{\int_0^{\frac{u'(c_H)}{v\tau}} a(\tilde{G}_H(v\tau t), s_H N_H(c_H, v))\, G'(t)\, dt}{\int_0^{\frac{v}{\tau}u'(c_F)} a(\tilde{G}_F(\frac{\tau}{v}t), s_F N_F(c_F, v))\, G'(t)\, dt} \frac{N_F\, G(\frac{v}{\tau}u'(c_F))}{N_H\, G(\frac{u'(c_H)}{v\tau})} \tag{40}$$

4.1.2 Equilibrium

Definition 2. An open-economy equilibrium consists of seven endogenous sequences $\{\Omega_{Ht}, \Omega_{Ft}, c_{Ht}, c_{Ft}, s_{Ht}, s_{Ft}, v_t\}_{t=0}^{\infty}$ which solve the following equations in each period:

I. Equation (18) for Home. In steady state, it can be used to eliminate Ω_H. With uncertainty, it links Ω_{Ht} to expectations about period $t+1$.

II. Equation (18) for Foreign. In steady state, it can be used to eliminate Ω_F. With uncertainty, it defines Ω_{Ft} to expectations about period $t+1$.

III. Equation (25) for Home for each t, replacing $\tilde{G}(\cdot)$ by (38) and $N(c)$ by (36). It links Ω_H, c_H, s_H, and v.

IV. Equation (25) for Foreign for each t, replacing $\tilde{G}(\cdot)$ by (39) and $N(c)$ by (37). It links Ω_F, c_F, s_F, and v.

V. Equation (30) for Home for each t, replacing $\tilde{G}(\cdot)$ and $N(c)$ as for (III.). It links Ω_H, c_H, s_H, and v.

VI. Equation (30) for Foreign for each t, replacing $\tilde{G}(\cdot)$ and $N(c)$ as for (IV.). It links Ω_F, c_F, s_F, and v.

VII. Equation (40), which links all seven endogenous variables for each t.

A numerical solution to this system involves six numerical integrals, and due to the curse of dimensionality, the complexity of this problem will increase significantly when allowing for free entry.

4.2 Many countries, with inverse-Pareto cost distributions

Suppose we assume two things: first, the cost distribution is inverse-Pareto, and second, the parameters are such that least productive firm is never active in equilibrium. Then the integral determining firm revenues can be evaluated exactly, which simplifies the free entry and balance of payments equations considerably. However, some generality is lost. For one, the profit share is exclusively determined by the Pareto shape parameter; in particular, it is independent of the reservation price and of consumer search effort.[10] For another, the distribution of costs among exporters, counting trade costs, is identical to that of domestic producers; thus, the average price charged by exporters and domestic producers is identical. These implications may be too strict for many applications.

Nevertheless, suppose that there are $n > 1$ countries, that there are N_i potential firms in country i, and that the cost distribution among potential firms is:

$$G_i(\phi) \equiv (z_i\phi)^\theta ,$$

where $\theta > 1$ is the shape parameter, and z_i is a productivity shifter for each country. We will maintain the assumption that $u'(c_i) < 1/z_i$ in equilibrium, so that the least efficient firm with $\phi = 1/z_i$ is never active. In that case, we can solve for the profit share of the economy (see Appendix A.5):

$$\Pi = \frac{1}{1+\theta}$$

And because conditional on being active in a market, producers from every source country have the same cost distribution (counting trade costs), all international revenue flows will be split into a fraction $1/(1+\theta)$ going to profits, and $\theta/(1+\theta)$ being used to pay for wages.

[10] This directly contradicts some intuitive results from the model where all firms have common costs. There, higher consumer search effort mitigates firm pricing power and reduces markups; here, this effect is exactly offset by the fact that sales are shifted towards high-markup firms. There, the optimal inflation rate depends on the costs of search effort and firm entry; here, it does not, because the equilibrium matching efficiency can be solved for independently of these costs.

Furthermore, it is not just the profit share that can be simplified, but the equations governing market equilibrium and optimal search, too. Consider the integrals in Equations (25) and (30). As not all firms are active in equilibrium, the support of $\tilde{G}$ is exactly $t \in [0, u'(c)]$; therefore, the value of the integral does not depend on the value of the bounds, but only on the shape of the distribution in between, and because this distribution is inverse-Pareto, the upper integration limit ceases to matter at all.[11] We can define the auxiliary functions:

$$\Phi(x) \equiv a(1,x) + \int_0^1 \left[-a_1(t^\theta, x)\right] \left[\theta t^{2\theta-2}\right] \, dt$$
$$\Psi(x) \equiv \; x \cdot \left[\, a(1,x)h(1,x) + \int_0^1 \left[-a_1(t^\theta, x)\right] h(t^\theta, x) \left[\theta t^{\theta-2}\right] \, dt \, \right]$$

4.2.1 Equilibrium

To summarize: suppose countries differ in terms of average firm productivity (z_i), disutility of searching (μ_i, measured relative to the disutility of working), money supply and expected money growth (M_i and γ_i). Suppose in every country the firm cost distribution and consumer search process have the same *shape*, and consumption sub-utility u is the same. Trade is subject to an iceberg cost τ; write $\tau_{ij} \equiv \tau$ if $i \neq j$ and $\tau_{ii} \equiv 1$. We can now define a steady-state open-economy equilibrium as follows:

Definition 3. For country $i = 1 \ldots n$, let c_i be consumption, s_i search effort, Ω_i the marginal value of money (the inverse of the wage), N_i the measure of potential firms domiciled in the country, and $\tilde{N}_i$ the measure of firms actively selling goods in the country (whether domiciled or not). For the country pair (i, j), let R_{ij} be the proportion of country i's expenditure (M_i) that is spent on goods from country j, and let v_{ij} be the wage-deflated exchange rate (defined in terms of country i's labor effort per country j's labor effort, so that an increase in v_{ij} is a depreciation of i's currency). Then, a steady-state open-economy

[11] Consequently, the equations would be unchanged if we used an imperfectly elastic model of the labor supply and the equilibrium wage, and worked with $\bar{p}w$ for the upper bound of costs, instead of substituting $w = 1/\Omega$ and $\bar{p}w = u'(c)$ as implied by perfectly elastic labor supply.

equilibrium without free entry is a list $\{c_i, s_i, \Omega_i, \tilde{N}_i, R_{ij}, v_{ij}\}_{i,j=1}^n$ satisfying the following equations for all countries and country pairs:

$$\begin{aligned}
\Omega_i &= \frac{\beta u'(c_i) c_i}{\gamma_i} \\
\frac{u'(c_i) c_i}{\Omega_i} &= \Phi(s_i \tilde{N}_i) \\
\frac{\mu_i s_i}{\Omega_i} &= \Psi(s_i \tilde{N}_i) \\
\tilde{N}_i &= [u'(c_i) z_i]^\theta \cdot \sum_{j=1}^n N_j (\tau_{ij} v_{ij})^{-\theta} \\
R_{ij} &= \frac{N_j (\tau_{ij} v_{ij})^{-\theta}}{\sum_{\ell=1}^n N_\ell (\tau_{i\ell} v_{i\ell})^{-\theta}} \\
v_{ij} &= \frac{\Omega_i}{\Omega_j} \frac{R_{ij}}{R_{ji}}.
\end{aligned}$$

If we are also interested in the nominal exchange rate, we can obtain it following Equation (35) by:

$$E_{ij} \equiv \frac{v_{ij} \Omega_j}{\Omega_i} \cdot \frac{M_i}{M_j}.$$

If we are interested in dynamic paths, we only need to replace the first equilibrium equation with Equation (18) for every country, and add a definition of how expectations are formed. Finally, we may be interested in firm entry:

Definition 4. Let κ_i be the cost of entry in country i, to be paid by each potential firm every period in terms of constant money units. (That is, the nominal payment is $M_i \kappa_i$.) Then a steady-state open-economy equilibrium with free entry is a list $\{c_i, s_i, \Omega_i, N_i, \tilde{N}_i, R_{ij}, v_{ij}\}_{i,j=1}^n$ which satisfies the same equations as the equilibrium without free entry, and in addition:

$$\kappa_i N_i = \frac{1}{1+\theta}$$

4.3 Comparative statics in the open economy

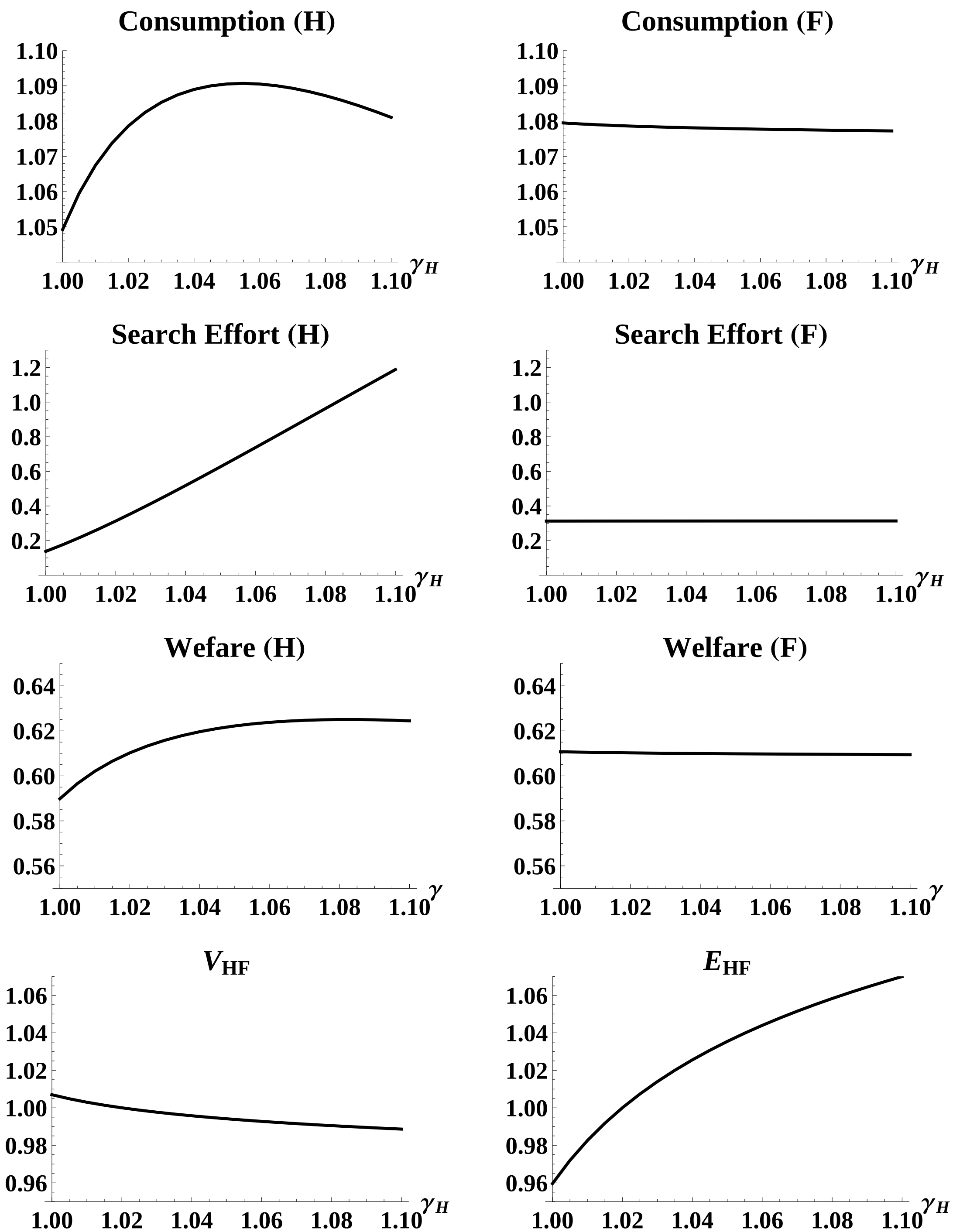
Similar to its effect in the closed economy, an increase in the Home search cost μ_H decreases both consumption and search effort at Home. Additionally, both the nominal and real exchange rate decrease (i.e. the Home currency appreciates). Foreign search effort is not affected, but Foreign consumption rises. The reason for this is that the lower consumption at home reduces the real wage at home, which increases foreign imports and causes the home currency to appreciate to restore trade balance.

Figure 4 shows that an increase in the Home money growth rate γ_H increases Home search effort strongly, and the effect on Home consumption is hump-shaped – again as in the closed economy. As a result, the effect of γ_H on Home welfare is hump-shaped, too, and there is an optimal level of γ_H. It is worth noting that the welfare-maximizing level of γ_H is higher than the consumption-maximizing level, even though (costly) search effort is higher there. Why? The reason is that by increasing search effort, less efficient firms are priced out of the market and production is shifted to the more efficient firms; this reduces Home labor demand and increases labor productivity (see appendix A.5).

What determines the level of the welfare-maximizing money growth rate? In general, it will be increasing in any parameter that reduces matching efficiency in equilibrium: search costs and firm entry costs (as in the closed economy), and also trade costs (because they make it harder for firms to sell abroad). However, the special case with inverse-Pareto costs is *very* special: neither trade costs, search costs, entry cost, or the TFP shifter z affect the welfare-maximizing growth rate. The only parameter that does is the shape of the cost distribution, θ, because this parameter alone determines the profit share and the shape of the Φ and Ψ functions. If θ is low, firm markups are high unless search effort is high, too; therefore, stimulating search effort with inflation is good for welfare. If θ is low, then most firms charge prices near marginal cost, and therefore the inflation tax effect dominates and the optimal money growth rate is low.

What are the effects of Home money growth on the Foreign economy? First, note that

Figure 4: **Effects of steady-state money growth in a two-country open economy;** on consumption, search effort, and welfare in both countries, plus the wage-deflated exchange rate (v) and the money-stock-deflated exchange rate (e).
The parameters are $\beta = 0.97$, $\sigma = 2$, $\mu = 0.02$, $z = 1$, $\theta = 4$, and $\kappa = .05$ in both countries; we hold $\gamma_F = 1.02$ fixed but γ_H varies. The firm cost distribution is $G(\phi) = (\phi \cdot z)^{\theta}$ (inverse-Pareto), the matching process is Logarithmic as defined in Appendix B, and the trade cost is $\tau = 1.3$. Welfare is measured in consumption equivalents: $W \equiv u^{-1}\left[u(c) - \mu s - L(s, c)\right]$.



the effect of γ_H on the various exchange rates is curious. In terms of money (or consumption goods), higher γ_H causes Home's currency to depreciate. However, in terms of labor costs, we see that Home's currency actually *appreciates*, as wages rise and Home labor effort falls. This is akin to a country with a monopoly position in foreign markets exploiting it by making its unions stronger and thereby restricting labor supply; the redistributive effect at Home washes out, but monopoly power abroad is exploited. Accordingly, we see that higher Home money growth causes Foreign consumption and welfare to fall, even as Foreign's currency appreciates. (Foreign search effort increases, because Foreign consumption falls (as long as $\sigma > 1$), but the effect is too small to be visible in the graph.)

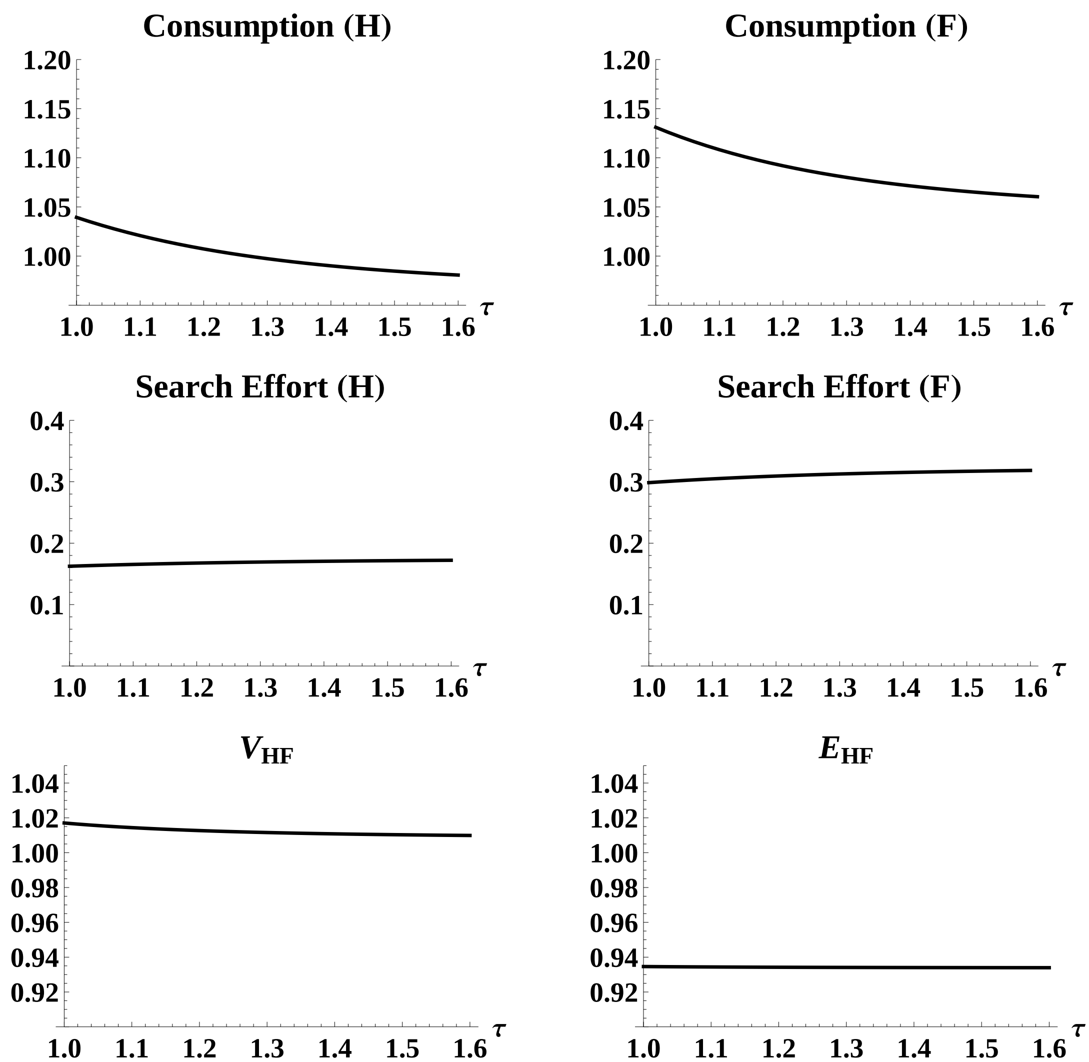
In the numerical example, there is no effect of money growth on firm entry in either country, but that is again an artifact of assuming that the cost distribution is inverse-Pareto – which makes the profit share a constant – and of assuming that entry costs are denominated in money rather than in labor or final goods. Relaxing these assumptions might be more realistic.

When search effort is fixed, which might be the correct model in the short run, the effects are very different. Higher Home money growth γ_H decreases Home consumption throughout, still decreases Foreign consumption, and decreases both the constant-money nominal and real exchange rates. Because they rely on the strong assumption of trade balance, these short-run exchange rate effects might be weaker in countries with well developed capital markets.

Figure 5, showing the comparative statics of the trade cost τ, shows that trade liberalization has the expected effects.[12] Lowering the trade cost increases consumption everywhere, and as long as $\sigma > 1$, the higher consumption causes lower search effort through an income effect. Firm entry is constant, but we already know that this is an artifact of the inverse-Pareto cost distribution. Just like in New Trade models, lower trade

[12] Note that in the figure, the countries are the same except for the cost of search effort, μ, which is twice as high at Home as in the Foreign country. Consequently, search effort and consumption are lower at Home, and the exchange rates do not equal 1.

Figure 5: **Effects of trade costs in a two-country asymmetric open economy;** on consumption and search effort in both countries, plus the wage-deflated exchange rate (v) and the money-stock-deflated exchange rate (e).
The parameters are $\beta = 0.97$, $\sigma = 2$, $z = 1$, $\theta = 4$, $\kappa = .05$, and $\gamma = 1.02$ in both countries; we set $\mu_H = 0.04$ but $\mu_F = 0.02$, so Home has a higher search cost than Foreign. The firm cost distribution is $G(\phi) = (\phi \cdot z)^{\theta}$ (inverse-Pareto), the matching process is Logarithmic as defined in Appendix B, and the trade cost is $\tau = 1.3$. Welfare is measured in consumption equivalents: $W \equiv u^{-1}\left[u(c) - \mu s - L(s, c)\right]$.



costs make it easier for exporters to compete abroad – increasing expected profits – and thereby make it harder for less efficient domestic firms to compete – decreasing expected profits. With inverse-Pareto costs, these effects exactly offset. With a general cost distribution, profits and entry could go in either direction. As we can see from the bottom row of the figure, trade liberalization also implies a slight depreciation of the currency of the less efficient country. Gains in consumption cause wages to increase in both countries (as long as $\sigma > 1$), which in turn increases prices; but prices increase by more in the less competitive market, which is the one with lower search effort.

5 Conclusion

In this paper, I have developed a monetary model with search frictions in the goods market, heterogeneous firms, and international trade. The model is amenable to studying the effect of monetary policy on markups, cost pass-through, firm entry, price dispersion, productivity, and welfare. More interesting labor and asset markets could be modeled as needed.

The result that the optimal inflation rate is increasing in the cost of search or the cost of firm entry is new and has important implications, because it implies that the effects of inflation in steady state will be different in countries with different structural characteristics. In related work, Herrenbrueck (2015) explores the implications of this observation: how countries' optimal inflation rates compare when they are integrated through trade, and whether countries should coordinate their inflation rates. That paper also estimates the costs and benefits of monetary convergence in the Eurozone prior to the introduction of the common currency.

References

Becker, S. and D. Nautz (2010). Inflation, price dispersion and market integration through the lens of a monetary search model. Discussion Papers 2010/2, Free University Berlin, School of Business & Economics.

Becker, S. S. (2011, March). What drives the relationship between inflation and price dispersion? market power vs. price rigidity. SFB 649 Discussion Papers SFB649DP2011-019, Humboldt University, Berlin, Germany.

Berentsen, A., G. Rocheteau, and S. Shi (2007, 01). Friedman meets hosios: Efficiency in search models of money. *Economic Journal 117*(516), 174–195.

Bernard, A. B., J. Eaton, J. B. Jensen, and S. Kortum (2003). Plants and productivity in international trade. *The American Economic Review 93*(4), 1268–1290.

Burdett, K. and K. L. Judd (1983, July). Equilibrium price dispersion. *Econometrica 51*(4), 955–69.

Burdett, K. and D. T. Mortensen (1998, May). Wage differentials, employer size, and unemployment. *International Economic Review 39*(2), 257–73.

Caglayan, M., A. Filiztekin, and M. T. Rauh (2008, October). Inflation, price dispersion, and market structure. *European Economic Review 52*(7), 1187–1208.

Craig, B. and G. Rocheteau (2008, 02). Inflation and welfare: A search approach. *Journal of Money, Credit and Banking 40*(1), 89–119.

Devereux, M. B. and S. Shi (2013). Vehicle currency. *International Economic Review 54*(1), 97–133.

Feenstra, R. C. (2003). A homothetic utility function for monopolistic competition models, without constant price elasticity. *Economics Letters 78*(1), 79–86.

Geromichalos, A. and K. M. Jung (2015). An over-the-counter approach to the forex market.

Geromichalos, A. and I. Simonovska (2014). Asset liquidity and international portfolio choice. *Journal of Economic Theory 151*, 342–380.

Head, A. and A. Kumar (2005, 05). Price dispersion, inflation, and welfare. *International Economic Review 46*(2), 533–572.

Head, A., A. Kumar, and B. Lapham (2010, 02). Market power, price adjustment, and inflation. *International Economic Review 51*(1), 73–98.

Head, A. and B. Lapham (2006, Dec). Search, market power, and inflation dynamics. 2006 Meeting Papers 559, Society for Economic Dynamics.

Head, A. and S. Shi (2003, October). A fundamental theory of exchange rates and direct currency trades. *Journal of Monetary Economics 50*(7), 1555–1591.

Herrenbrueck, L. (2015). Optimal monetary policy, international coordination, and currency unions.

Hopenhayn, H. A. (1992). Entry, exit, and firm dynamics in long run equilibrium. *Econometrica: Journal of the Econometric Society*, 1127–1150.

Li, V. E. (1995, November). The optimal taxation of fiat money in search equilibrium. *International Economic Review 36*(4), 927–42.

Liu, L. Q., L. Wang, and R. Wright (2011). On the hot potato effect of inflation: Intensive versus extensive margins. *Macroeconomic Dynamics 15*(S2), 191–216.

Melitz, M. J. (2003, November). The impact of trade on intra-industry reallocations and aggregate industry productivity. *Econometrica 71*(6), 1695–1725.

Mortensen, D. T. (2005, 05). A comment on "price dispersion, inflation, and welfare" by a. head and a. kumar. *International Economic Review 46*(2), 573–578.

Shi, S. (1997, January). A divisible search model of fiat money. *Econometrica 65*(1), 75–102.

Shi, S. (2006, March). A microfoundation of monetary economics. Working Papers tecipa-211, University of Toronto, Department of Economics.

Simonovska, I. (2015). Income differences and prices of tradables: Insights from an online retailer. *The Review of Economic Studies 82*(4), 1612–1656.

Wang, L. (2014). Endogenous search, price dispersion, and welfare. Technical report.

Zhang, C. (2014). An information-based theory of international currency. *Journal of International Economics 93*(2), 286–301.

A Appendix: derivations and proofs

Proof of Lemma 2.1: (Based on equation (34) in Burdett and Mortensen (1998).) We can rank

$$\pi(p_1;\phi_1) \ \geq \ \pi(p_2;\phi_1) \ > \pi(p_2;\phi_2) \ \geq \ \pi(p_1;\phi_2) \tag{41}$$

The first and last inequality follow from profit maximization. The middle inequality follows from $\phi_1 < \phi_2$. Subtracting the fourth term from the first, and the third term from the second, and rearranging, we get

$$\frac{p_2}{a(p_2)} \ \geq \ \frac{p_1}{a(p_1)} \tag{42}$$

As $F(p)$ is a continuous c.d.f. with connected support, it is strictly increasing on its support. From definition (2), it is easy to see that $a(F)$ is a strictly decreasing function of F. Therefore, $a(F(p))$ is a strictly decreasing function of p on the support of F. □

Proof of Lemma 2.2: $p(\phi)$ solves the FOC. Regarding the boundary condition: either $\bar{p}$ is binding for some sellers, in which case the marginal seller will charge $\bar{p}$ and make zero profits. Or the least efficient seller is still able to compete, but must charge the highest price by the ranking condition; then, however, no buyer with another choice will buy from this seller. Therefore, a buyer who is willing to buy from this seller will accept any price up to $\bar{p}$, and the least efficient seller will charge $\bar{p}$. □

A.1 Derivation: market equilibrium

Throughout this section, I ignore the number of firms and let s stand for the more proper sN or $sN(c)$.

Consider the case where all firms have the same cost ϕ. Inverting equation (10), we can solve for the price a firm with rank $F \in [0, 1]$ charges:

$$p(F,S) = \left[\frac{1}{\phi w} - \left(\frac{1}{\phi w} - \frac{1}{\bar{p}}\right)\frac{a(1,S)}{a(F,S)}\right]^{-1}, \tag{43}$$

when S is the aggregate search effort in the economy. Then, a household that searches with effort s, can expect the following consumption:

$$c(s,S) = \int_0^1 \frac{a(F,s)}{p(F,S)}\, dF \tag{44}$$

Substituting (43), imposing symmetry $s = S$, and substituting (19) for the wage and (7) for the reservation price, we obtain:

$$\begin{aligned} c &= \int_0^1 \left[\frac{\Omega}{\phi}a(F,s) - \left(\frac{\Omega}{\phi} - \frac{\Omega}{u'(c)}\right)a(1,s)\right] dz \\ &= \Omega\left(\frac{1-a(1,s)}{\phi} + \frac{a(1,s)}{u'(c)}\right), \end{aligned}$$

using $\int_0^1 a(F,s)dF = 1$. The result, equation (21), follows.

When firms have heterogeneous costs, we use the price function (17) in (44). We obtain, using symmetry $s = S$:

$$\begin{aligned} c &= \int_{\underline{\phi}}^{\bar{p}/w} \frac{a(\tilde{G}(\phi),s)}{p(\phi)}\tilde{G}'(\phi)\, d\phi \\ &= \int_{\underline{\phi}}^{\bar{p}/w} \left[\frac{a(1,s)}{\bar{p}} - \int_{\phi}^{\bar{p}/w} \frac{a_1(\tilde{G}(t),s)}{wt}\tilde{G}'(t)\, dt\right]\tilde{G}'(\phi)\, d\phi \\ &= \frac{a(1,s)}{\bar{p}} - \int_{\underline{\phi}}^{\bar{p}/w}\left[\int_{\underline{\phi}}^{t}\tilde{G}'(\phi)\, d\phi\right]\frac{a_1(\tilde{G}(t),s)}{wt}\tilde{G}'(t)\, dt \\ &= \frac{a(1,s)}{\bar{p}} - \int_{\underline{\phi}}^{\bar{p}/w}\frac{a_1(\tilde{G}(t),s)}{wt}\tilde{G}(t)\tilde{G}'(t)\, dt. \end{aligned}$$

Substituting $\bar{p}$ and w, equation (25) follows.

A.2 Derivation: optimal search

Return to equation (44). The condition for optimal search is $u'(c)\frac{\partial c}{\partial s} = \mu$, and it yields a symmetric equilibrium when households are identical in their money holdings M and their search cost μ, as long as $u(c(s,S))$ is strictly concave in s for any S.[13] Then, assuming all firms have the same cost ϕ:

$$\left.\frac{\partial c}{\partial s}\right|_{s=S} = \int_0^1 \frac{a_2(F,s)}{p(F,s)}\,dF$$
$$= \int_0^1 \frac{a_2(F,s)}{w\phi}\,dF - \left(\frac{1}{\phi w} - \frac{1}{\bar{p}}\right) a(1,s) \int_0^1 \frac{a_2(F,s)}{a(F,s)}\,dF$$
$$= -\left(\frac{1}{\phi w} - \frac{1}{\bar{p}}\right) a(1,s)h(1,s)$$

using the definition of $h(F,s)$ (27), and the fact that $\int_0^1 a(F,s)dF = 1$ and therefore $\int_0^1 a_2(F,s)dF = 0$. Substituting $\bar{p}$ and w, we obtain:

$$\mu = u'(c(s,S))\left.\frac{\partial c}{\partial s}\right|_{s=S}$$
$$= -u'(c)\left(\frac{\Omega}{\phi} - \frac{\Omega}{u'(c)}\right) a(1,s)h(1,s).$$

Equation (28) follows, once we account for the number of firms which is assumed to multiply s. Equation (29) is obtained by dividing equation (28) by equation (21) in order to eliminate Ω, which expresses the optimal search decision in a purely contemporaneous equation.

When firms have heterogeneous costs, we again use the price function (17) in (44). We obtain, using symmetry $s = S$:

$$\left.\frac{\partial c}{\partial s}\right|_{s=S} = \int_{\underline{\phi}}^{\bar{p}/w} \frac{a_2(\tilde{G}(\phi),s)}{p(\phi)}\tilde{G}'(\phi)\,d\phi$$

[13] A formal analysis of this condition is hard and does not currently exist in the literature. However, I have verified numerically that the condition holds for the geometric matching process.

$$= \int_{\underline{\phi}}^{\bar{p}/w} \left[\frac{a(1,s)}{\bar{p}} - \int_{\phi}^{\bar{p}/w} \frac{a_1(\tilde{G}(t),s)}{wt} \tilde{G}'(t)\, dt \right] \frac{a_2(\tilde{G}(\phi),s)}{a(\tilde{G}(\phi),s)} \tilde{G}'(\phi)\, d\phi$$
$$= \frac{a(1,s)h(1,s)}{\bar{p}} - \int_{\underline{\phi}}^{\bar{p}/w} \frac{a_1(\tilde{G}(t),s)}{wt} h(\tilde{G}(\phi),s)\, \tilde{G}'(t)\, dt.$$

Substituting $\bar{p}$ and w, we obtain:

$$\mu = u'(c) \left(\frac{\Omega}{u'(c)} a(1,s)h(1,s) - \int_{\underline{\phi}}^{u'(c)} \frac{\Omega}{t}\, a_1(\tilde{G}(t),s)\, h(\tilde{G}(\phi),s)\, \tilde{G}'(t)\, dt \right).$$

Equation (30) follows.

A.3 Equilibrium

Lemma A.1. *The functions* $a(F,s) = \sum_{k=1}^{K} q_k(s)k(1-F)^{k-1}$ *and* $h(F,s) = \int_0^F \frac{a_2(z,s)}{a(z,s)} dz$ *have the following properties:*

(i) $a(F,s)$ *is strictly decreasing in* F *for all* $s > 0$, *and* $a(F,0) \equiv 1$.

(ii) $h(1,s) < 0$ *for* $s > 0$, *and* $h(1,0) = 0$.

(iii) $a(1,s)h(1,s)$ *is zero for* $s = 0$, *is negative for* $s > 0$, *and converges to 0 from below as* $s \to \infty$.

Proof. For (i), note that each component of the sum is strictly decreasing in F, and the coefficients $q_k(s)k$ are positive. By assumption M2, $q_1(0) = 1$, which implies that $q_k(0) = 0$ for all $k \geq 2$, therefore $a(F,0) \equiv 1$. For (ii), $a(F,0) \equiv 1$ implies that:

$$h(1,0) = \int_0^1 a_2(z,0)\, dz = \frac{\partial}{\partial s} \left[\int_0^1 a(z,s)\, dz \right]_{s=0} = \frac{\partial}{\partial s}(1) = 0.$$

And by assumption M3, the partial derivative $a_2(F,s)$ is positive for low F and negative for high F (indeed, $\int_0^1 a_2(z,0)\, dz = 0$). But (i) implies that dividing by $a(F,s)$ in the integral in $h(1,s)$, more weight is put on high values of F and therefore on the negative parts of $a_2(F,s)$. Assertion (ii) follows.

Concerning (iii), $a(1,s)h(1,s)$ is zero for $s = 0$, and negative for any s by (ii) and by the fact that $a(1,s) = q_1(s)$ must be strictly positive for any $s \geq 0$. Then, for $\varepsilon > 0$,

$$\begin{aligned}
(a(1,s)h(1,s))^2 &= \left(\int_0^1 \frac{a(1,s)}{a(z,s)} a_2(z,s)\, dz\right)^2 \\
&\leq \int_0^1 \left(\frac{a(1,s)}{a(z,s)}\right)^2 dz \cdot \int_0^1 (a_2(z,s))^2\, dz \\
&\leq \int_0^1 (a_2(z,s))^2\, dz \\
&= \int_0^1 \left(\sum_{k=1}^{K} q_k'(s) k\, (1-z)^{k-1}\right)^2 dz \\
&\leq \int_0^1 \left(\sum_{k=1}^{K} k^{2+\varepsilon} \left(q_k'(s)\right)^2 \cdot \sum_{k=1}^{K} k^{-\varepsilon} (1-z)^{2k-2}\right) dz \\
&= \sum_{k=1}^{K} k^{2+\varepsilon} \left(q_k'(s)\right)^2 \cdot \sum_{k=1}^{K} \frac{k^{-\varepsilon}}{2k-1}.
\end{aligned}$$

The first and third inequalities are applications of the Cauchy-Schwarz inequality. The second follows from fact (i), which implies that $a(1,s) \leq a(F,s)$ for all $F \in [0,1]$. What remains is a term that converges to zero as $s \to \infty$ by assumption M6. The proof suggests that assumption M6 could be weakened if $K < \infty$. □

Full proof of theorem 3.1: Consider equation (21). If $\Omega < u'(c)c$, as assumed (E3), it has a solution relation in (s,c)-space with $s > 0$ and $c > 0$. And $c \to 0$ implies $s \to 0$, as $a(1,s) = q_1(s) < 1$ with equality if and only if $s = 0$. Re-write the equation as:

$$c = \Omega \left(\frac{1 - a(1,s)}{\phi} - \frac{a(1,s)}{u'(c)}\right). \tag{45}$$

By assumption E2, $u'(c)$ is increasing if $c \to 0$, and more quickly than c decreases itself. The term $\frac{a(1,s)}{u'(c)}$ is dominated and can be ignored when analyzing the equation in the neighborhood of $(0,0)$. What is left is the relationship $c \approx \frac{\Omega}{\phi}(1 - a(1,s))$, which implies the implicit derivative $\frac{dc}{ds}\big|_{s\to 0, c\to 0} = \frac{\Omega}{\phi}(-q_1'(s))$. But the derivative $-q_1'(s)$ is bounded as $s \to 0$ (by assumption M5 on the matching process). On the other hand, consider the equation

(28). In the neighborhood of $(0,0)$, its implicit derivative is infinite, i.e. it approaches the c-axis.

Now, let c^* be the efficient level of consumption given by $(u')^{-1}(\phi)$. Equation (21) must approach $c \to c^*$, whether for $s \to 0$ or $s \to \infty$. However, the graph of equation (28) achieves a maximum $\hat{c} < c^*$ in (s, c)-space: by lemma A.1(iii), $-a(1, s)h(1, s)$ is positive and inverse u-shaped, and by assumption E2, the graph approaches the corners $(s \to 0, c \to 0)$ and $(s \to \infty, c \to 0)$. Consequently, the graphs of equation (21) and (28) must cross exactly once.

The steady-state analysis is slightly easier: use equation (22) to divide out $\frac{u'(c)}{\phi}$ from equation (29). The result is an equation defining a negative relationship between s and c:

$$\mu = (u'(c)c - \Omega)\,\frac{-a(1,s)h(1,s)}{1-a(1,s)}$$

$$\Rightarrow \qquad \frac{\gamma}{\gamma-\beta}\mu = u'(c)c\frac{-a(1,s)h(1,s)}{1-a(1,s)}. \tag{46}$$

(The same as equation (48).) The graph of this equation is strictly decreasing, meets the c-axis, and either converges to or meets the s-axis. As the graph of (22) is upward sloping and approaches $c \to 0$ as $s \to 0$ and $c \to c^*$ as $s \to \infty$, the two graphs must cross exactly once. □

A.4 Comparative statics and the effects of inflation

Lemma A.2. *Assume E1 and E2 from theorem 3.1. An increase in the production cost ϕ leads to a fall in consumption and a rise in search effort. An increase in the disutility of search, μ, leads to a fall in both consumption and search effort. With free entry, an increase in the cost of firm entry, κ, decreases consumption and has an ambiguous effect on search effort.*

Proof. First, let the number of firms be fixed. Consider the version of the model where all firms have the same cost ϕ, that is, equations (22), and (29). The search cost μ only affects the second equation, while ϕ affects both. But ϕ enters each equation as $u'(c)/\phi$, which we can therefore divide out of the second one. We obtain two equations in (s, c)-space:

$$\frac{\gamma}{\beta} = \frac{u'(c)}{\phi}(1 - a(1, sN)) + a(1, sN) \tag{47}$$

$$\mu = N\, u'(c)c \left(1 - \frac{\beta}{\gamma}\right) \frac{-a(1, sN)h(1, sN)}{1 - a(1, sN)} \tag{48}$$

Equation (47) is identical to equation (22), which we already know slopes up in (s, c)-space. Furthermore, the curve shifts right/down if ϕ increases. Equation (48) is new but easy to understand. The term $\frac{-a(1,sN)h(1,sN)}{1-a(1,sN)}$ is strictly decreasing in s, and if we assume that $u'(c)c$ is strictly decreasing in c, then the equation describes a downward-sloping curve in (s, c)-space. This curve shifts left/down if μ increases, towards the origin. Consequently, the effect of an increase in the production cost ϕ is a fall in consumption and a rise in search effort. On the other hand, an increase in the disutility of search, μ, leads to a fall in both consumption and search effort.[14]

If there is free entry, dividing equation (29) by equation (31) yields a curve in (sN, c)-space that takes the role of equation (29) in the previous discussion. Equation (21) and the new curve depend only on sN, not on s or N directly, and the new curve has the same shape as (29). So an increase in either μ or κ implies lower consumption and lower levels of sN. The zero-profits condition (31) implies that N increases if sN decreases, so search effort unambiguously falls. On the other hand, N decreases in κ according to zero-profits, so the net effect on search effort is ambiguous in principle. For the parameters considered in this paper, the effect reducing search effort seems to dominate. □

Lemma A.3. *In a steady state of a SMSE, the Friedman rule $\gamma = \beta$ implies $c = 0$, $\gamma > \beta$ implies $c > 0$ and $\gamma \to \infty$ implies $c \to 0$. Therefore, the consumption-maximizing rate of money growth satisfies $\gamma > \beta$.*

Proof. Use the market equilibrium equation (22) and the contemporaneous optimal search

[14] If $u'(c)c$ was constant, however, the curve (48) would be vertical, and if $u'(c)c$ was increasing in c, it would be downward sloping. In that case, an increase in ϕ would also reduce both consumption and search effort, provided equilibria exist for both the old and the new ϕ.

equation (29) (or (26) and (30) with heterogeneous costs). The market equilibrium curve slopes upwards, crosses the optimal search curve exactly once, and shifts right as money growth γ increases: faster money growth reduces the value of money, which creates market power. Constant consumption would therefore require ever-increasing search effort. As $\gamma \to \beta$, it becomes Γ-shaped and converges to the c-axis on the left and the $c = c^*$-line on top (in case of common costs, with $u'(c^*) = \phi$); as $\gamma \to \infty$, it converges to the s-axis. The graph of the contemporaneous optimal search equation (29), on the other hand, has an inverted u-shape in (s, c)-space, and does not depend on γ (or any other expectations), so it contains the set of possible equilibria. It approaches $c \to 0$ both for $s \to 0$ and $s \to \infty$, because of assumptions E2, M2, and M3, and achieves a maximum $\hat{c}$ for $s > 0$. As a result, increasing money growth *traces out* the optimal search curve, and achieves the maximum $\hat{c}$ for $\gamma > \beta$. Higher money growth also reduces profits, which implies exit of firms, and vice versa. This tends to dampen, but not reverse, the effects of money growth on consumption and search. □

Lemma A.4. *Let* $\mu_2 > \mu_1$*.*

(i) Say that the graph of the contemporaneous optimal search equation (29) *achieves the maximum* $\hat{c}$ *for* $\hat{s}$*. Then* $\hat{c}_2 < \hat{c}_1$ *and* $\hat{s}_2 > \hat{s}_1$*, that is, the maximum point shifts down and to the right in* (s, c)*-space.*

(ii) The consumption-maximizing rate of money growth is higher for μ_2 *than for* μ_1*.*

Proof. For (i), consider equation (29):

$$\mu = N\, u'(c)c\, \frac{-a(1, sN)h(1, sN)\left(\frac{u'(c)}{\phi} - 1\right)}{\frac{u'(c)}{\phi}(1 - a(1, sN)) + a(1, sN)}$$

Conjecture that higher μ implies lower c due to the $u'(c)c$-term. Thus, $\frac{u'(c)}{\phi}$ is lower, too, so s must rise a bit to make up for it. As a result, the entire curve shifts down and just a bit right. For (ii), note that the market equilibrium curve (22) is upward sloping in (s,c)-space. Therefore, if money growth was optimal for μ_1, it must cross the new curve (29) to the left of both the new and the old $\hat{s}$, and higher money growth is necessary to achieve $(\hat{s}_2, \hat{c}_2)$. □

A.5 Derivation: profits, productivity, markups

Define productivity φ to be consumption divided by labor input. When all firms have the same marginal cost ϕ, it is easy to check that $\varphi = 1/\phi$. Profits are given by (31).

When firms have heterogeneous costs, firm profits are given by (9). Using the policy function $p(\phi)$ of (17), and defining $\Gamma(\phi) = \int_{\underline{\phi}}^{\phi} t\, \tilde{G}'(t)\, dt$ (the conditional mean below ϕ), the aggregate profit level is:

$$\begin{aligned}
\Pi &= \int_{\underline{\phi}}^{u'(c)} \pi(\phi)\, \tilde{G}'(\phi)\, d\phi \\
&= \int_{\underline{\phi}}^{u'(c)} \left(1 - \frac{\phi}{\Omega p(\phi)}\right) a(\tilde{G}(\phi), sN(c))\, \tilde{G}'(\phi)\, d\phi \\
&= 1 - \int_{\underline{\phi}}^{u'(c)} \left[\frac{\phi}{u'(c)} a(1, sN(c)) - \int_{\phi}^{u'(c)} \frac{\phi}{t} a_1(\tilde{G}(\phi), sN(c)) \tilde{G}'(t)\right] \tilde{G}'(\phi)\, d\phi \\
&= 1 - a(1, sN(c)) \frac{\Gamma(u'(c))}{u'(c)} - \int_{\underline{\phi}}^{u'(c)} \frac{\Gamma(t)}{t} \left(-a_1(\tilde{G}(t), sN(c))\right) \tilde{G}'(t)\, dt. \qquad (49)
\end{aligned}$$

A firm with cost ϕ and constant-money price p will sell $1/p$ units of output per consumer, and will therefore have a labor use of $L(\phi) = \frac{\phi}{p} a(F(p), sN(c))$. Integrating over all firms, it is easy to see the similarity with the previous calculation for profits. Indeed, each firms' profits can be expressed as revenue—wage*labor, and since total revenue is 1 (in constant-money terms), total labor L satisfies: $\Pi + wL = 1$, or:

$$L = \Omega\,(1 - \Pi)\,, \qquad (50)$$

with Π as above or as in (32).

Since we defined productivity $\varphi = c/L$, we have the full expression (reintroducing $N(c)$):

$$\varphi(s,c) = \frac{a(1, sN(c))\frac{\Gamma(u'(c))}{u'(c)} + \int_{\underline{\phi}}^{u'(c)} \frac{\Gamma(t)}{t}\left(-a_1(\tilde{G}(t), sN(c))\right)\tilde{G}'(t)\,dt}{a(1, sN(c)) + \int_{\underline{\phi}}^{u'(c)} \frac{u'(c)}{t}\left(-a_1(\tilde{G}(t), sN(c))\right)\tilde{G}(t)\tilde{G}'(t)\,dt}, \tag{51}$$

which is not directly dependent on expectations, hence easy to calculate given an equilibrium solution (s, c).

If we define a firm's markup as its Lerner index, $\frac{p-\phi w}{p}$, then the average markup is equal to the share of money (or output) that is retained as profits. There are two ways to see this. First, define the aggregate price level to be $P = M/c$. Nominal costs of production are $Mw\frac{L}{c}$ (recall that w is the constant-money wage, so Mw is the nominal wage, and c/L is productivity), and we can define the aggregate markup as $\frac{P - Mw\frac{L}{c}}{P} = 1 - wL = \Pi$.

Alternatively, we can define each firm's markup as $\frac{p(\phi)-w\phi}{p(\phi)}$. We can integrate across firms, weighting by the sales of each firm:

$$\begin{aligned}\text{markup} &= \int_{\underline{\phi}}^{u'(c)} \left(1 - \frac{w\phi}{p(\phi)}\right) a(\tilde{G}(\phi), s)\,\tilde{G}'(\phi)\,d\phi \\ &= \Pi.\end{aligned}$$

Accordingly, we can take total constant-money profits (i.e. the profit share of the economy) as a measure of market power, which is easier to calculate than if the markup were to be defined as $\frac{p}{\phi w}$.

Finally, suppose that the cost distribution is inverse-Pareto with shape $\theta > 0$. Also suppose that the upper bound on the cost distribution never binds, so that the upper bound on the distribution of active firms is $\bar{p}w$. That is, the cost distribution of active firms is $\tilde{G}(\phi) = (\phi/(\bar{p}w))^{\theta}$, and thus the Γ-function satisfies:

$$\frac{\Gamma(\phi)}{\phi} = \frac{\theta}{1+\theta}\tilde{G}(\phi)$$

Plugging these results into Equation (49) yields:

$$\begin{aligned}\Pi &= 1 - \frac{\theta}{1+\theta}a(1, sN(c)) - \frac{\theta}{1+\theta}\int_{\underline{\phi}}^{\bar{p}w}\left(-a_1(\tilde{G}(t), sN(c))\right)\tilde{G}(t)\tilde{G}'(t)\,dt \\ &= 1 - \frac{\theta}{1+\theta}a(1, sN(c)) - \frac{\theta}{1+\theta}\left[1 - a(1, sN(c))\right],\end{aligned}$$

using integration by parts, and the result $\Pi = 1/(1+\theta)$ follows.

Notice that for this result we do not need any particular model of wages; the fact that $\tilde{G}(\bar{p}w) = 1$ is enough. In the model described above, labor supply is perfectly elastic and therefore we have $\bar{p}w = u'(c)$. But given any other labor supply function, we will still obtain $\Pi = 1/(1+\theta)$ as long as (a) the cost distribution is inverse-Pareto with shape θ, and (b) the firm with the highest cost is never active.

B Appendix: Matching

The assumptions on the matching process $Q(k|\eta)$ sufficient for the results in this paper are:

M1. $Q(k|\eta)$ is a cumulative distribution function with support $\{1 \ldots K\}$, where $K \in \{2 \ldots \infty\}$. $q_k(\eta)$ is used to denote the probability mass function.

M2. $q_1(\eta) \in (0, 1)$ for all $\eta > 0$, and $q_1(0) = 1$.

M3. When $\eta' > \eta$, $Q(k|\eta') < Q(k|\eta)$: the distribution of quotes with higher search effort strictly first order stochastically dominates the one with lower search effort.

M4. $Q(k|\eta)$ is continuously differentiable in η (which implies that the p.m.f. $q_k(\eta)$ is continuously differentiable in η, too).

M5. $q_1'(\eta)$ is bounded as $s \to 0$.

M6. There exists an $\varepsilon > 0$ such that the series $\sum_{k=1}^{K} k^{2+\varepsilon} \left(q_k'(\eta)\right)^2$ converges for any $\eta > 0$, and that the series limit approaches 0 as $\eta \to \infty$.

A particularly tractable family of matching processes can be derived in the following way. Start by assuming that the number of quotes k follows a negative binomial distribution on $k \in \{0, 1, \ldots\}$, parametrized in the usual way with "number of failures" $r > 0$ and "success probability" $p \in (0, 1)$. (In this interpretation, r is an integer but as we see below, the extension to non-integer values of r is smooth.) In that case, the probability mass function is:

$$q_k = \frac{\Gamma(k+r)}{\Gamma(k+1)\Gamma(r)}(1-p)^r p^k$$

The mean number of quotes is $\bar{k} = rp/(1-p)$. Here, a more convenient parametrization is using the unconditional mean $\eta \equiv \bar{k}$ and a dispersion parameter $\rho \equiv 1/r$ to replace (r, p).[15] We obtain the following probability mass function:

$$q_k = \frac{\Gamma(k+1/\rho)}{\Gamma(k+1)\Gamma(1/\rho)}(1+\rho\eta)^{-1/\rho}\left(\frac{\rho\eta}{1+\rho\eta}\right)^k$$

Using the definitions of the transactions function J and the arrival function a, we can compute them in closed form:

$$\begin{aligned} J(F) &= \sum_{k=1}^{\infty} \frac{q_k}{1-q_0}\left(1-(1-F)^k\right) \\ &= \frac{1-(1+\rho\eta F)^{-1/\rho}}{1-(1+\rho\eta)^{-1/\rho}}, \end{aligned}$$

which clearly satisfies $J(0) = 0$ and $J(1) = 1$, and:

[15] This parametrization is often used in econometrics when estimating count data.

$$
\begin{aligned}
a(F) &= \sum_{k=1}^{\infty} \frac{q_k}{1-q_0}\, k\,(1-F)^{k-1} \\
&= \frac{\eta}{1-(1+\rho\eta)^{-1/\rho}}\,(1+\rho\eta F)^{-1-1/\rho}\,,
\end{aligned}
$$

which is clearly positive and decreasing, as required. (Because these functions are defined as proportions of those matched, they are automatically conditional on $k > 0$, so for all of the cases discussed below, whether we allow $k = 0$ only matters for a household's decisions and not for those of the firms.) A very nice feature is that this solution to the matching process extends more generally than $\rho \in (0, \infty)$. We can distinguish four special cases:

Binomial: Let $\rho < 0$ and $\eta \in (0, -1/\rho)$. If $-1/\rho$ is a positive integer, then the number of quotes follows a binomial distribution with maximum $K = -1/\rho$, success probability $-\rho\eta$, and mean η. A subcase is when $K = 2$ (shoppers have either zero, one, or two quotes): this is the most commonly used case in the literature building on Burdett and Judd (1983). It has an appealing foundation (Head and Kumar, 2005): if shoppers could choose the probability weights q_k subject to linear costs of receiving quotes, they would choose to mix between q_1 and q_2 only. But for the purposes in this paper, negative ρ is inconvenient: household consumption becomes a *convex* function of search effort, requiring a mixed strategy equilibrium for the choice of search effort. The other cases (which put more weight on receiving large numbers of quotes) do not have this problem.

Poisson: Taking the limit as $\rho \to 0$, for fixed $\eta > 0$, the number of quotes follows a Poisson distribution with intensity η (Mortensen, 2005; Head and Lapham, 2006), and the arrival function becomes exponential: $a(F) \propto \exp(-\eta F)$.

Negative Binomial: As discussed in the derivation, when $\rho > 0$ and $\eta > 0$, the number of quotes follows a negative binomial distribution with number of failures $1/\rho$ and success probability $\rho\eta/(1 + \rho\eta)$.

Logarithmic: Taking the limit as $\rho \to \infty$, the probability q_0 of receiving no quotes at all converges to 1 for any fixed mean. So the distribution only converges to a well-defined limit if we condition on $k > 1$ and if η is rescaled appropriately. In that case, the limit is the discrete logarithmic distribution $q_k = \left(\frac{\eta}{1+\eta}\right)^k \frac{1}{\log(1+\eta)}/k$, with mean η and $k \in \{1, \ldots, \infty\}$.[16] The associated transactions functions are:

$$J(F) = \frac{\log(1+\eta F)}{\log(1+\eta)}$$
$$a(F) = \frac{\eta}{\log(1+\eta)} (1+\eta F)^{-1}$$

Because of its scope, I refer to this family of matching processes as the **generalized binomial** family. [17] It has one property that could be very useful for empirical applications: if $\rho > 0$, the distribution of customers per firm is a truncated Pareto distribution with shape parameter $\rho/(1+\rho)$, and the ratio of the upper to the lower bound of the distribution is $a(0)/a(1) = (1+\rho\eta)^{1+1/\rho}$.

Specifically, as $a(F)$ is the proportion of customers that buy from a firm with position F in the price distribution, the inverse of $a(F)$ describes the fraction of firms charging a lower price than a firm with $a(F)$ customers, which is also the fraction of firms with more customers than $a(F)$. If we allow $q_0 > 0$ (some searchers go unmatched), the CDF of the *number* of customers per firm is:

$$F(x) = \frac{1+\rho\eta}{\eta} - \frac{1}{\rho\eta}\left(\frac{x}{\eta}\right)^{-\rho/(1+\rho)}.$$

[16] To distinguish this distribution from the continuous logarithmic distribution, some authors call it the "log-series distribution" because it can be derived from the power series of the logarithm.

[17] The fact that the region $\rho \in (-\infty, -1/\eta)$ is excluded has an intuitive interpretation, too. It corresponds to the restriction in Burdett and Judd (1983) that shoppers must receive more than one quote with positive probability; otherwise, all firms will charge the reservation price, and due to the lack of price dispersion, no equilibrium with positive search intensity exists.